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THE SMOOTH WHITNEY FIBERING CONJECTURE
AND
OPEN BOOKS IN WHITNEY AND BEKKA STRATIFICATIONS

C. MUROLO, A. du PLESSIS, D.J.A. TROTMAN

Using continuous controlled liftings of vector fields, we first prove for Bekka's (c)- and hence Whitney (b)-regular stratifications $\mathcal{X}$ that near every point of a stratum X with $depth_\Sigma(X) = 1$ there exists a local $C^{0,1}$ foliation. Then we construct a local open book structure near each point of X and use this result to prove the general smooth version of the Whitney fibering conjecture near every point of an arbitrary stratum X of $\mathcal{X}$. As a consequence we improve the Thom-Mather regularity of the local trivialization maps of a proper stratified submersion $f : \mathcal{X} \to M$ into a manifold.

1. Introduction. In his famous paper of 1965 **[Wh]** H. Whitney proposed a local fibering property around points of a complex analytic variety. More precisely he conjectured that every complex analytic variety V admits a stratification such that a neighbourhood U of each point is fibered by copies of the intersection of U with the stratum M containing the point. He asked also that the fibers be holomorphic manifolds and that their tangent spaces vary continuously as nearby points approach X (see section 2 for a precise formulation).

Note that if one does not require the continuity of tangent spaces to the fibers then the Thom-Mather isotopy theorem **[Th]**, $\mathbf{[Ma]}_{1,2}$ suffices to prove a smooth version of Whitney's conjecture.

In 1989 R. Hardt and D. Sullivan gave a proof of a similar conclusion for holomorphic varieties but again without the essential continuity of the tangent spaces to the fibers **[HS]**.

From 1993 the first author studied the possibility of obtaining the analogous property in the case of smooth real stratified spaces in his thesis under the direction of the third author who conjectured this property be true for Whitney (b)-regular stratifications.

The solution of the smooth version of the Whitney conjecture was necessary to use the notion of *semidifferentiability* introduced in $\mathbf{[Mu]}_1$ and $\mathbf{[MT]}_4$ with the aim of obtaining the preservation of regularity of a substratified space of a stratification after a deformation by stratified isotopy $\mathbf{[MPT]}_{1,2}$, useful in showing the conjectured representation of homology by Whitney stratified cycles $\mathbf{[Go]}_{1,2}$ (an open problem since 1981) and also in approaching the unsolved conjecture of Thom on the existence of Whitney triangulations and cellulations of Whitney stratified sets (two final chapters of the thesis remained in a manuscript form $\mathbf{[MT]}_{5,6}$). It improves moreover the Thom-Mather isotopy theorem by ensuring a *horizontally-*C^1 regularity ($\mathbf{[MT]}_{3,4}$, §5 Theorems 10 and 11).

The first and third authors began a collaboration on this research with A. du Plessis, whose book of 1995 **[PW]** with C. T. C. Wall introduced the notion of E-tame retractions, as retractions whose fibers are foliations having an analogous regularity property. More precisely, with the aim of proving that multi-transversality with respect to a given partition in submanifolds of a jet space is a sufficient condition for strong C^0-stability du Plessis and Wall introduced and studied various regularity conditions for retractions $r : M \to N$

between two smooth manifolds : the *Tame, Very tame and Extremely tame retractions.* These last, the E-tame retractions, were characterized by the fact that the foliations defined by their fibres are of class $C^{0,1}$. This property in a stratified context for a local "horizontal" retraction $\pi' : \pi_X^{-1}(U_{x_0}) \to \pi_X^{-1}(x_0)$ is equivalent to a real $C^{0,1}$ version of the conclusion of the Whitney fibering conjecture ([**MT**]$_4$ §4.3). Concrete situations where these tame retractions exist were studied by du Plessis and Wall [**PW**], and Feragen [**Fe**] who found particular cases where retractions can be glued.

In 2007, P. Berger, in his Ph.D. thesis supervised by J.-C. Yoccoz, with the aim of generalizing some fundamental results of Hirsch-Pugh-Shub on dynamical systems in higher dimensions, needed the smooth version of the Whitney fibering conjecture to study the persistence of stratifications of normally expanded laminations [**Ber**].

In 2014, A. Parusinski and L. Paunescu [**PP**] constructed for a given germ of complex or real analytic set a stratification satisfying a strong trivialization (called *arc-wise analytic*) property along each stratum and then proved the Whitney fibering conjecture in the real and complex, local analytic and global algebraic cases.

In this paper we prove a result which implies a smooth version of Whitney's conjecture. This is that any Bekka (c)-regular stratification satisfies a smooth version of the Whitney properties.

Recall that (c)-regularity is strictly weaker than Whitney's (b)-regularity [**Be**], and that every complex analytic variety admits a (b)-regular stratification (hence (c)-regular) [**Wh**]. More generally every subanalytic set admits a (b)-regular stratification [**Hi**], [**DW**], [**Ha**], [**LSW**], as does every definable subset in an o-minimal structure [**VM**], [**Loi**], [**NTT**]. Thus the smooth version of Whitney's property holds for these classes of sets.

The contents of the paper are as follows.

In section 2, we present the Whitney fibering conjecture as stated in the original paper of H. Whitney [**Wh**].

In section 3, we review the most important classes of regular stratifications: the abstract stratified sets of Thom and Mather [**Th**], [**Ma**]$_{1,2}$, Whitney (b)-regular stratifications [**Wh**], and the (c)-regular stratifications of Karim Bekka [**Be**], and we briefly recall the relations between them and the first isotopy theorem which holds for them.

In section 4, we introduce the notion of nice foliation on a stratification by two strata $X < Y$ in $\mathbb{R}^n$ and prove a theorem for gluing these foliations whose methods will be frequently used in the proof of our first main Theorem 3.

In section 5, we first recall in §5.1 some important properties of the stratified topological triviality map obtained by using continuous canonical lifted frame fields [**MT**]$_{2,3,4}$. In §5.2 we recall some useful properties of the frame fields tangent to the horizontal leaves of this trivialization map. Then in §5.3 we prove a two strata version of our main result, Theorem 3 (and Theorem 4) proving that the smooth version of the Whitney fibering conjecture holds for every stratum X with $depth_\Sigma(X) = 1$ of a (c)-regular stratification.

In section 6 under the same hypotheses as Theorems 3 and 4 we construct a local open book structure for Bekka (c)-regular (Theorem 5) and Whitney (b)-regular (Theorem 6) stratifications for every stratum X with $depth_\Sigma(X) = 1$. These results will play an important role in the proof of our main theorem in section 7.

In section 7 we prove our main theorems. First we use the notion of conical chart (Definition 10) and the local open book structure of section 6 to prove the conclusions of the smooth Whitney fibering conjecture for any stratum X of a (c)-regular stratification

$\mathcal{X} = (A, \Sigma)$ having arbitrary depth (Theorem 7). Then we use Theorem 7 to extend Theorems 5 and 6 of section 6 to a stratum of arbitrary depth (Theorem 8).

In section 8 we apply Theorem 7 to the results of **[MT]**$_{1,3,4}$, where we introduced the notions of horizontally-C^1 stratified controlled morphism $f : \mathcal{X} \to \mathcal{X}'$ (Definition 11) to prove that the flows of the continuous lifted vector fields to a stratum X have a horizontally-C^1 regularity, stronger than C^0-regularity, but weaker than C^1-regularity (Corollaries 5 and 6) and we deduce a horizontally-C^1 version of Thom's 1^{st} Isotopy Theorem for a stratified proper submersion $f : \mathcal{X} \to M$ into a manifold (Theorems 8 and 9). Then using the finer notion of $\mathcal{F}$-semidifferentiability (Definition 12), we improve these results by stating an $\mathcal{F}$-semidifferentiable version of Thom's Isotopy Theorem which extends the horizontally-C^1 convergence of the topological trivialisation of f to all points of the strata Y' such that $X \leq Y' \leq Y$ (Theorem 10).

We thank Dennis Sullivan for drawing our attention to Whitney's original fibering conjecture and to his own work with Bob Hardt on this conjecture **[HS]**. We thank also Edmond Fedida, Etienne Ghys, Pierre Molino, David Spring and the late Bill Thurston for useful discussions.

2. The Whitney fibering conjecture.

In his famous article *Local Properties of Analytic Varieties* **[Wh]**, after introducing the well-known (a) and (b)-regularity conditions and showing that *"Every (real or complex) analytic variety V admits a (b)-regular stratification"*, H. Whitney gave the following definition:

Definition 1. A stratification Σ of an analytic variety V will be considered *"good"* if each point $p_0 \in V$ admits a neighbourhood U_0 in V having a foliation $\mathcal{H}_{p_0} = \{F(q)\}_q$ obtained in the following way. Let M be the stratum of Σ containing p_0, $M_0 := M \cap U_0$ and $N_0 := (T_{p_0}M)^{\perp} \cap U_0$ (where $\perp$ means the orthogonal complement in the ambient space).

Then U_0 is homeomorphic to $M_0 \times N_0$ through a map $\phi : M_0 \times N_0 \to U_0$, $\phi = \phi(p, q)$ satisfying the following properties:

i) ϕ is analytic in $p \in M_0$ and continuous in $q \in N_0$;
ii) $\mathcal{H}_{p_0} = \{F(q)\}_q$ is exactly the foliation $\{M_q := \phi(M_0 \times \{q\})\}_{q \in N_0}$ induced by ϕ;
iii) every restriction $\phi_{|M_0 \times q} : M_0 \times \{q\} \to F(q)$ to a leaf of $\mathcal{H}_{p_0}$ is a biholomorphism;
iv) both restrictions $\phi_{|M_0 \times \{q_0\}} = id_{M_0}$ and $\phi_{|\{p_0\} \times N_0} = id_{N_0}$ are the identity.

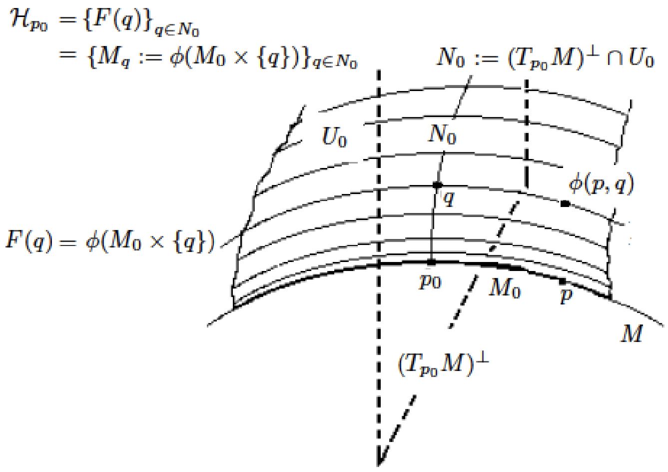

Figure 1

Whitney's definition of 1965 was a *precursor* of the idea that suitable regular stratifications have the property of local topological triviality, an idea completely clarified in the years 1969-70 by the famous Thom-Mather first isotopy theorem **[Ma]**$_{1,2}$ **[Th]**.

Whitney called such a map ϕ a *semianalytic fibration (for Σ) near p_0* and remarked that an analytic variety V does not have (in general) a stratification admitting near each point an analytic fibration. He gave the celebrated counterexemple (the *four lines family*)

$$V := \{(x,y,z) \in \mathbb{C}^3 \mid xy(y-x)\big(y-(3+t)x\big) = 0\ \}$$

and stated the following conjecture:

Whitney fibering conjecture. *"Every analytic variety V has a stratification admitting in each point $p_0 \in V$ a semianalytic fibration".*

Whitney comments furthermore that *". . . a stratification satisfying the conjecture (possibly with further conditions on ϕ) would probably be sufficient for all needs".*

Remark 1. Whitney also states in a commentary that every stratification Σ of V with such a semianalytic fibration near a point p_0 is automatically (a)-regular at all points of the neighbourhood $U_0 \cap M$ of p_0 in M, because the properties of ϕ imply the convergence of the tangent planes to the leaves of the foliation $\mathcal{H}_{p_0} = \{F(q)\}_{q \in N_0}$:

$$(L_p): \qquad \lim_{z \to p} T_z F(z) = T_p M\ .$$

So, for Whitney, (a)-regularity is a consequence of the existence of such a *semi-analytic fibration.* In fact he wrote that in a local analysis in which M_0 is identified with the $(x_1,\ldots,x_d)$-plane ($d = \dim M$), for each stratum $M_j > M_0$, *"any fiber $F(q)$, with $q \in M_j$ sufficiently near to p_0 is near $F(p_0) = M_0$ .... and $F(q)$ is expressed by holomorphic functions $x_i = f_i(x_1,\ldots,x_d)$, $i = 1,\ldots,n-d$. These functions are small throughout M_0; <u>hence their partial derivatives are small in a smaller neighbourhood of p_0</u>.*
Since $F(q) \subseteq M_j$ if $q \in M_j$, this clearly implies the condition (a)".

However, this argument is not valid in general as A. du Plessis explained in a conference in 2005 at the CIRM.

Definition 2. Because the limit condition $(L_p): \lim_{z\to p} T_zF(z) = T_pM$ will be very important for us we have redefined it in a more general C^1-real context as the *(a)-regularity of a local horizontal foliation* **[Mu]**$_1$, **[MT]**$_4$ and (by abuse of language) we will refer to it as the *C^1 version of the Whitney fibering conjecture* : in this paper we do not seek any (re)-stratification but our aim is to prove the conclusions of the original conjecture of Whitney for each point x of every stratum X of a fixed arbitrary Bekka (c)-regular stratification $\mathcal{X}$ with C^1 strata.

Let us mention some work on the fibering conjecture.

Whitney proved (**[Wh]**, §12) that the conjecture holds for every analytic hypersurface V of $\mathbb{C}^n$ for all points in $(n-2)$-strata after restratification of V.

Later, in 1983, Hardt **[Ha]** indicated a possible solution of the problem in the real analytic case and in 1988, Hardt and Sullivan **[HS]** treated the problem for complex algebraic varieties. The conclusion obtained by Hardt and Sullivan **[HS]** is weaker than that proposed by Whitney, in particular they did not obtain the condition $\lim\limits_{z\to p} T_zF(z) = T_pM$, i.e. the (a)-regularity of the foliation $\mathcal{H}_{p_0}$.

Very recently, in 2015, A. Parusinski and L. Paunescu **[PP]**, using a slightly stronger version of Zariski equisingularity constructed for a given germ of complex or real analytic set, a stratification satisfying a strong (real arc-analytic with respect to all variables and analytic with respect to the parameter space) trivialization property along each stratum (the authors call such trivializations *arc-wise analytic*). Then using a generalization of Whitney Interpolation they prove the Whitney fibering conjecture in the real and complex, local analytic and global algebraic cases.

We conclude this section by recalling the following globalization problem ([**Wh**] §9):

Problem. *"May one fibre a complete neighbourhood of any stratum ?"*
That is :
Can one find a global stratified foliation of a complete neighbourhood of any stratum ?

We will not deal with this problem, but we just remark that without restratifying the smaller stratum it cannot have a solution in general. In fact, as P. Berger wrote to us (see his Ph.D. Thesis p. 60 or [**Ber**] p. 38) by considering the stratification of two strata $M = S^2 \times \{(0,0)\} < S^2 \times (\mathbb{R}^2 - \{(0,0)\})$ of $S^2 \times \mathbb{R}^2$, a global foliation of a neighbourhood of $S^2 \times \{(0,0)\}$ in $S^2 \times (\mathbb{R}^2 - \{(0,0)\})$ cannot exist, because otherwise, starting from an arbitrary non-zero vector tangent to S^2 one could define by holonomy a continuous non-zero vector field on the whole of S^2 which cannot exist.

3. Bekka (c)-regular stratified spaces.

We recall that a *stratification* of a topological space A is a locally finite partition Σ of A into C^1 connected manifolds (called the *strata* of Σ) satisfying the *frontier condition* : if X and Y are disjoint strata such that X intersects the closure of Y, then X is contained in the closure of Y. We write then $X < Y$ and $\partial Y = \overline{Y} - Y$ so that $\overline{Y} = Y \sqcup \big(\sqcup_{X<Y} X\big)$ and $\partial Y = \sqcup_{X<Y} X$ ($\sqcup =$ disjoint union).

The pair $\mathcal{X} = (A, \Sigma)$ is called a *stratified space* with *support* A and *stratification* Σ. The union of the strata of dimension $\leq k$ is called the *k-skeleton*, denoted by A_k, inducing a stratified space $\mathcal{X}_k = (A_k, \Sigma_{|A_k})$.

A *stratified map* $f : \mathcal{X} \to \mathcal{X}'$ between stratified spaces $\mathcal{X} = (A, \Sigma)$ and $\mathcal{X}' = (B, \Sigma')$ is a continuous map $f : A \to B$ which sends each stratum X of $\mathcal{X}$ into a unique stratum X' of $\mathcal{X}'$, such that the restriction $f_X : X \to X'$ is smooth. We call such a map f a *stratified homeomorphism* if f is a global homeomorphism and each f_X is a diffeomorphism.

A stratified vector field on $\mathcal{X}$ is a family $\zeta = \{\zeta_X\}_{X\in\Sigma}$ of vector fields, such that ζ_X is a smooth vector field on the stratum X.

Extra conditions may be imposed on the stratification Σ, such as to be an *abstract stratified set* in the sense of Thom-Mather [**GWPL**], [**Ma**]$_{1,2}$, [**Ve**] or, when A is a subset of a C^1 manifold, to satisfy conditions (a) or (b) of Whitney [**Ma**]$_{1,2}$, [**Wh**], or (c) of K. Bekka [**Be**].

Definition 3. (Thom and Mather) Let $\mathcal{X} = (A, \Sigma)$ be a stratified space.

A family $\mathcal{F} = \{(\pi_X, \rho_X, T_X)\}_{X\in\Sigma}$ is called *a system of control data)* for $\mathcal{X}$ if for each stratum X we have that:

1) T_X is a neighbourhood of X in A (called a *tubular neighbourhood of* X);
2) $\pi_X : T_X \to X$ is a continuous retraction of T_X onto X (called *projection on* X);
3) $\rho_X : T_X \to [0, \infty[$ is a continuous function such that $X = \rho_X^{-1}(0)$ (called the *distance function from* X)

and, furthermore, for every pair of adjacent strata $X < Y$, by considering the restriction maps $\pi_{XY} = \pi_{X|T_{XY}}$ and $\rho_{XY} = \rho_{X|T_{XY}}$ on the subset $T_{XY} = T_X \cap Y$, we have that :

5) the map $(\pi_{XY}, \rho_{XY}) : T_{XY} \to X \times [0, \infty[$ is a smooth submersion (it follows

in particular that $\dim X < \dim Y$);

6) for every stratum Z of $\mathcal{X}$ such that $Z > Y > X$ and for every $z \in T_{YZ} \cap T_{XZ}$ the following *control conditions* are satisfied :
i) $\pi_{XY}\pi_{YZ}(z) = \pi_{XZ}(z)$ (called the π-*control condition*),
ii) $\rho_{XY}\pi_{YZ}(z) = \rho_{XZ}(z)$ (called the ρ-*control condition*).

In what follows we will pose $T_X(\epsilon) = \rho_X^{-1}([0,\epsilon))\,, \forall\, \epsilon \geq 0$, and without loss of generality will assume $T_X = T_X(1)$ $[\mathbf{Ma}]_{1,2}$, [**GWPL**].

If A is Hausdorff, locally compact and admits a countable basis for its topology, the pair $(\mathcal{X}, \mathcal{F})$ is called an *abstract stratified set.* Since one usually works with a unique system of control data of $\mathcal{X}$, in what follows we will omit $\mathcal{F}$.

If $\mathcal{X}$ is an abstract stratified set, then A is metrizable and the tubular neighbourhoods $\{T_X\}_{X\in\Sigma}$ may (and will always) be chosen such that: "$T_{XY} \neq \emptyset$ if and only if $X < Y$, or $X > Y$ or $X = Y$" (see $[\mathbf{Ma}]_1$, page 41 and following).

Let $f : \mathcal{X} \to \mathcal{X}'$ be a stratified map between two abstract stratified sets and fix two systems of control data $\mathcal{F} = \{(T_X, \pi_X, \rho_X)\}_{X\in\Sigma}$ and $\mathcal{F}' = \{(T_{X'}, \pi_{X'}, \rho_{X'})\}_{X'\in\Sigma'}$ respectively of $\mathcal{X}$ and $\mathcal{X}'$. The map f is called *controlled (with respect to $\mathcal{F}$ and $\mathcal{F}'$)* if when $X < Y$ there exists $\epsilon > 0$ such that for all $y \in T_{XY}(\epsilon) = T_X(\epsilon) \cap Y$ the following two *control conditions* hold :

$$\begin{cases} \pi_{X'Y'} f_Y(y) = f_X \pi_{XY}(y) & \text{(the } \pi\text{-}\textit{control condition} \text{ for } f) \\ \\ \rho_{X'Y'} f_Y(y) = \rho_{XY}(y) & \text{(the } \rho\text{-}\textit{control condition} \text{ for } f). \end{cases}$$

Similarly, a stratified vector field $\zeta = \{\zeta_X\}_{X\in\Sigma}$ is *controlled (with respect to $\mathcal{F}$)* if the following two *control conditions* hold:

$$\begin{cases} \pi_{XY*}(\zeta_Y(y)) = \zeta_X(\pi_{XY}(y)) & \text{(the } \pi\text{-}\textit{control condition} \text{ for } \zeta) \\ \\ \rho_{XY*}(\zeta_Y(y)) = 0 & \text{(the } \rho\text{-}\textit{control condition} \text{ for } \zeta). \end{cases}$$

The notion of system of control data of $\mathcal{X}$, introduced by Mather in $[\mathbf{Ma}]_{1,2}$, is the fundamental tool allowing one to obtain good extensions of vector fields.

In fact, we have $[\mathbf{Ma}]_{1,2}$, [**GWPL**] :

Proposition 1. *If $\mathcal{X}$ is an abstract stratified set with C^2 strata, every vector field ζ_X defined on a stratum X of $\mathcal{X}$ admits a stratified (π,ρ)-controlled lifting $\zeta_{T_X} = \{\zeta_Y\}_{Y\geq X}$ defined on a tubular neighbourhood T_X of X.*

Moreover, if ζ_X admits a global flow $\{\phi_t : X \to X\}_{t\in\mathbb{R}}$, then such a lifting ζ_{T_X} admits again a global flow $\{\phi_{T_X\,t} : T_X \to T_X\}_{t\in\mathbb{R}}$, and every $\phi_{T_X\,t} : T_X \to T_X$ is a stratified, continuous and (π,ρ)-controlled homeomorphism. $\square$

Definition 4. (K. Bekka 1991). A stratified space (A,Σ) in $\mathbb{R}^n$ is called (c)-*regular* if, for every stratum $X \in \Sigma$, there exists an open neighbourhood U_X of X in $\mathbb{R}^n$ and a C^1 function $\rho_X : U_X \to [0,\infty[$, such that $\rho_X^{-1}(0) = X$, and such that its stratified restriction to the *star* of $\mathcal{X}$:

$$\rho_X \;\; : \;\; Star(X) \cap U_X \to [0,\infty[\qquad \text{is a Thom map,}$$

where $Star(X) = \cup_{Y\in\Sigma\,,\,Y\geq X} Y$ and the stratification on $Star(X) \cap U_X$ is induced by Σ.

The (c)-regularity of Bekka states exactly that for every pair of adjacent strata $X < Y$, the tangent spaces at $y \in Y$ to the level hypersurfaces $\rho_X^{-1}(\epsilon)$ (where $\epsilon = \rho_X(y)$) have limits which contain T_xX when $y \to x \in X$.

Remark 2. A Bekka (c)-regular stratified space $\mathcal{X} = (A, \Sigma)$ admits a system of control data $\{(\pi_X, \rho_X, T_X)\}_{X\in\Sigma}$ in which for each stratum $X \in \Sigma$, $T_X = U_X \cap A$, and π_X, ρ_X are restrictions of C^1 maps defined on U_X **[Be]**. Thus (c)-regular stratifications admit a structure of abstract stratified set and so Proposition 1 holds for them.

We underline moreover that in this case, for each vector field ζ_X on a stratum X of A, the stratified (π, ρ)-controlled lifting $\zeta_{T_X} = \{\zeta_Y\}_{Y\geq X}$ defined on a tubular neighbourhood T_X of X may be chosen *continuous* **[Be]**, **[Pl]**, $[\mathbf{MT}]_{2,3}$, $[\mathbf{Sh}]_{1,2}$; this gives a more regular lifted flow $\{\phi_{T_X,t} : T_X \to T_X\}_{t\in\mathbb{R}}$).

We recall now the most important properties of lifting of vector fields on such regular stratifications and the most useful relations between them :

i) the condition "to be a Thom-Mather abstract stratified set" implies the existence of controlled lifting of vector fields $[\mathbf{Ma}]_{1,2}$;

ii) Bekka's (c)-regularity is *characterized* by the existence of (π, ρ)-controlled and *continuous* lifting of vector fields **[Be]**, **[Pl]**, $[\mathbf{MT}]_{2,3}$, $[\mathbf{Sh}]_{1,2}$, and implies the property *"to be a Thom-Mather abstract stratified set"* $[\mathbf{Ma}]_{1,2}$. Moreover (c)-regular stratifications admit systems of control data whose maps $\{(\pi_X, \rho_X) : T_X \to X \times [0, \infty[\,\}_X$ are C^1 **[Be]**. Bekka's (c)-regular stratifications have been used notably in **[Si]** to prove a Poincaré-Hopf index theorem for radial stratified vector fields and in **[Nad]** to study Morse theory and tilting sheaves on Schubert stratifications.

Finally we recall the following important facts :

a) (b)-regularity implies (c)-regularity **[Be]**, $[\mathbf{Tr}]_1$;

b) every abstract stratified set admits a (b)-regular embedding **[Na]**, **[Te]**, and even **[No]** a subanalytic (w)-regular and hence (b)-regular **[Kuo]**, **[Ve]** embedding in some $\mathbb{R}^N$;

c) abstract stratified sets admit triangulations, smooth in the sense of Goresky **[Go]**;

d) the first isotopy theorem of Thom-Mather holds for all the kinds of stratification considered above, using the (claimed) properties of stratified lifting of vector fields.

The first isotopy theorem of Thom-Mather applied to a projection map $\pi_X : T_X \to X$ on the stratum X can be stated as follows :

Theorem 1. *Let $\mathcal{X} = (A, \Sigma)$ be an abstract stratified closed subset in $\mathbb{R}^n$ with C^2-strata, X a stratum of $\mathcal{X}$ and $x_0 \in X$ and U_{x_0} a domain of a chart near x_0 of X.*

For every frame field $(v_1(x), \ldots, v_l(x))$ of U_{x_0} ($l = \dim X$) having a global flow, the (π, ρ)-controlled liftings $(v_1(z), \ldots, v_l(z))$ on $\pi_X^{-1}(U_{x_0})$ have global flows $(\phi_1, \ldots, \phi_l)$ and the map

$$\begin{array}{rcl} H = H_{x_0} \ : \ U_{x_0} \times \pi_X^{-1}(x_0) & \longrightarrow & \pi_X^{-1}(U_{x_0}) \\ (t_1, \ldots, t_l, z_0) & \longmapsto & \phi_l(t_l, \ldots, \phi_1(t_1, z_0)\ldots) \end{array}$$

is a stratified homeomorphism, a diffeomorphism on each stratum of $U_{x_0} \times \pi_X^{-1}(x_0)$. $\square$

4. Gluing nice foliations by generating frame fields.

The proof of a smooth version of the Whitney fibering conjecture for (c)-regular stratifications having two strata $X < Y$ that we will give in section 5 (Theorem 3) needs a careful analysis of the properties of the local foliations $\mathcal{H}_x$ induced by a topological trivialization H obtained using *continuous* (π, ρ)-controlled lifting of vector fields $[\mathbf{MT}]_4$ from

X to Y. It is moreover strongly based on a careful gluing of foliations locally defined on neighbourhoods of each $y \in Y$ in which the tangent planes to the foliations are very close to the canonical distribution $\mathcal{D}_X$.

So, with the aim of reproducing the essential properties and situations of the local foliations $\mathcal{H}_x$, in this section we introduce the notion of *nice* l-dimensional foliation of a k-submanifold Y of $\mathbb{R}^n$ with $X = \mathbb{R}^l \times \{0^{n-l}\} < Y$ and we prove Theorem 2 allowing us to glue together nice foliations using adapted partitions of unity.

Although the statements of this section are not directly used in the proof of Theorem 4, the methods of the proof used in the gluing in Theorem 3, Corollary 1 and Remark 3 contain all the basic ingredients that allow us to prove the smooth version of the Whitney fibering conjecture for (c)-regular stratifications. Thus we include this short section.

Definition 5 (Nice Foliation). Let $X < Y$ be a (c)-regular stratification in $\mathbb{R}^n$ with $X = \mathbb{R}^l \times \{0^m\}$ $(m = n - l)$, with $\rho_X : T_X(1) \to [0, 1[$ a distance function from X such that $\lim_{y \to x} T_y \rho_{XY}^{-1}(\rho_{XY}(y))) \supseteq T_x X$ and $\pi_X = p : \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}^l \times \{0^m\}$ the canonical projection.

Let $\mathcal{F} = \{M_\alpha\}_\alpha$ be an l-dimensional foliation of $T_{XY}(\epsilon)$ for an $\epsilon > 0$.

We say that $\mathcal{F} = \{M_\alpha\}_\alpha$ is a *nice l-foliation* of $T_{XY}(\epsilon)$ if each leaf of $\mathcal{F}$ is contained in a unique level hypersurface $\rho_{XY}^{-1}(\epsilon)$ of Y and the restriction of p, $p_{M_\alpha} : M_\alpha \to p(M_\alpha)$, to each leaf M_α of $\mathcal{F}$ is a C^1-diffeomorphism on $X = \mathbb{R}^l \times \{0^m\}$.

In this case, for every $z \in M_\alpha$, if $x = p(z)$, M_α is transverse to the affine space $p^{-1}(x) = \{x\} \times \mathbb{R}^m$ and $p^{-1}(x) \cap M_\alpha = \{z\}$.

Denoting $M_z := M_\alpha$ the leaf of $\mathcal{F}$ containing z, the foliation $\mathcal{F}$ can be reparametrised by $\mathcal{F} := \{M_z\}_{z \in p^{-1}(x)}$ for every $x \in X = \mathbb{R}^l \times \{0^m\}$.

Moreover for every $i = 1, \ldots, l$ the standard constant vector field $E_i(x) = E_i$ of X lifts to a unique vector field $w_i(z)$ tangent to M_z which is $w_i(z) = p_{*z}^{-1}(E_i) \cap T_z M_z = q_{M_z * x}(E_i)$ where $q_{M_z} : X \to M_z$ denotes the inverse map of p_{M_z}.

If $\mathcal{D}$ is a distribution of l-planes tangent to Y and transverse to ker $p_{|Y*}$ (thus each $p_{Y*y} : \mathcal{D}(y) \to X$ is an isomorphism) for every $\delta > 0$ we say that $\mathcal{F}$ *is δ-close* to $\mathcal{D}$ if

$$|| \, w_i(y) - v_i(y) \, || \; \leq \; \delta \qquad \forall \, y \in T_{XY}(\epsilon) \quad \text{ and } \quad \forall \, i = 1, \ldots, l$$

where $(v_1, \ldots, v_l)$ denote the *canonical* liftings of the $(E_1, \ldots, E_l)$ on $\mathcal{D}$.

We will apply this notion to the canonical distribution $\mathcal{D}_X$ which is continuous on X.

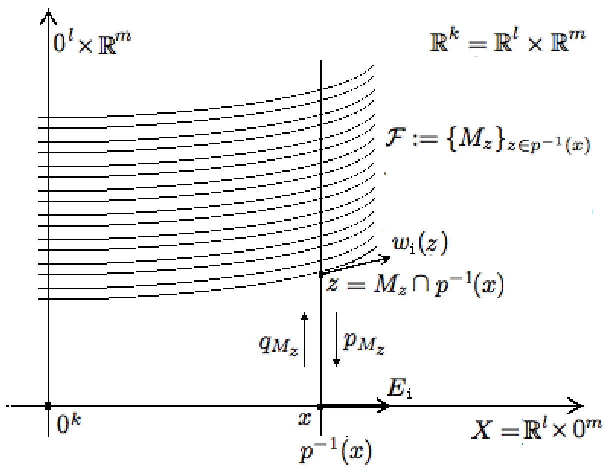

Figure 2

Definition 6. Let $\mathcal{F} = \{M_z\}_{z\in p^{-1}(x)}$ be a nice l-foliation of $T_{XY}(\epsilon)$.
We will say that *a frame field* $(w_1,\dots,w_l)$ *of* $T_{XY}(\epsilon)$ *generates* $\mathcal{F}$ if :

i) $(w_1,\dots,w_l)$ is tangent to $\mathcal{F}$, hence $\forall\, y\in M_z$, $T_yM_z = [(w_1(y),\dots,w_l(y)]$;
ii) $p_{M_z*y}(w_i(y)) = E_i$, for every $y\in T_{XY}(\epsilon)$ and $i=1,\dots,l$;
iii) all Lie brackets $[w_i,w_j]=0$, for every $i\neq j=1,\dots,l$.

In particular the liftings $\left(w_i(z) := q_{M_z*x}(E_i)\right)_{i=1,\dots,l}$ generate $\mathcal{F}$.

If $(w_1,\dots,w_l)$ generates a nice foliation $\mathcal{F}$, it defines an integrable l-distribution $\mathcal{D}(y) = [w_1(y),\dots,w_l(y)]$ of $T_{XY}(\epsilon)$ tangent to $\mathcal{F}$ and whose integral l-manifolds are exactly the leaves of $\mathcal{F}$.

If $\psi_1,\dots,\psi_l$ denote respectively the flows of $w_1,\dots,w_l$, and H the map defined by

$$H:\mathbb{R}^l\times p^{-1}(x)\to T_{XY}(\epsilon) \quad , \quad H(t_1,\dots,t_l,z_0) = \psi_l(t_l,\dots,\psi_1(t_1,z_0))\dots)\,,$$

then the leaves of $\mathcal{F}$ are the images $M_z = H(\mathbb{R}^l\times\{z\})$, one has $\mathcal{F} = \left\{H(\mathbb{R}^l\times\{z\})\right\}_{z\in p^{-1}(x)}$ and moreover $w_i(y) = H_{*y}(E_i)$ for every $i=1,\dots,l$.

Theorem 2. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two nice l-foliations of $T_{XY}(\epsilon)$ generated respectively by frame fields $(w_1^1,\dots,w_l^1)$ and $(w_1^2,\dots,w_l^2)$ such that $w_l^1 = w_l^2$ and let $a<b\in\mathbb{R}$.*

Then $\mathcal{F}_1$ and $\mathcal{F}_2$ can be glued in a nice l-foliation $\mathcal{F} := \mathcal{F}_1\vee\mathcal{F}_2$ of $T_{XY}(\epsilon)$ such that :

$$\mathcal{F}_1\vee\mathcal{F}_2 \;=\; \begin{cases} \mathcal{F}_1 \quad on \quad U_1 \;:=\; T_{XY}(\epsilon)\cap(]-\infty,a\,[\times\mathbb{R}^{n-1}) \\ and \\ \mathcal{F}_2 \quad on \quad U_2 \;:=\; T_{XY}(\epsilon)\cap(]\,b,+\infty[\times\mathbb{R}^{n-1})\,. \end{cases}$$

Moreover, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are both δ-close to a distribution $\mathcal{D}$ then $\mathcal{F}$ is δ-close to $\mathcal{D}$ too.

Proof. We will define $\mathcal{F}$ through a generating frame field $(w_1,\dots,w_l)$ which we will construct by a decreasing induction. We start by defining the vector field $w_l := w_l^1 = w_l^2$.

For $i=l-1$ take a partition of unity $\{\alpha,\beta:\mathbb{R}\to[0,1]\}$ subordinate to the open covering $\left\{\,]-\infty,b\,[,\,]\,a,+\infty[\,\right\}$ of $\mathbb{R}$ and extend it to a partition of unity of $T_{XY}(\epsilon)$, $\mathcal{P}_{l-1} = \{\alpha_{l-1}(y),\beta_{l-1}(y)\}$ subordinate to the open covering $\{U_1,U_2\}$ of $T_{XY}(\epsilon)$ which is constant along the trajectories of $w_l = w_l^1 = w_l^2$ (we call it *adapted* to $\{U_1,U_2\}$).

Then defining the vector field :

$$w_{l-1}(y) \;:=\; \alpha_{l-1}(y)w_{l-1}^1(y) \;+\; \beta_{l-1}(y)w_{l-1}^2(y)$$

one finds :

$$\begin{cases} p_{Y*y}(w_{l-1}(y)) = \alpha_{l-1}(y)p_{Y*y}(w_{l-1}^1(y)) + \beta_{l-1}(y)p_{Y*y}(w_{l-1}^2(y)) = \alpha_{l-1}(y)\cdot E_1 + \beta_{l-1}(y)\cdot E_1 = E_1 \\ \rho_{XY*y}(w_{l-1}(y)) = \alpha_{l-1}(y)\rho_{XY*y}(w_{l-1}^1(y)) + \beta_{l-1}(y)\rho_{XY*y}(w_{l-1}^2(y)) = \alpha_{l-1}(y)\cdot 0 + \beta_{l-1}(y)\cdot 0 = 0. \end{cases}$$

Moreover, the Lie bracket $[w_{l-1},w_l]$ satisfies :

$$\begin{aligned} [w_{l-1}(y),w_l(y)] \;&=\; [\alpha_{l-1}w_{l-1}^1(y),w_l(y)] \;+\; [\beta_{l-1}w_{l-1}^2(y),w_l(y)] \\ &= \Big(\alpha_{l-1\,*y}(w_l(y))\cdot w_{l-1}^1(y) \quad + \quad \alpha_{l-1}(y)[w_{l-1}^1(y),w_l(y)]\Big) \;+ \\ &+ \Big(\beta_{l-1\,*y}(w_l(y))\cdot w_{l-1}^2(y) \quad + \quad \beta_{l-1}(y)[w_{l-1}^2(y),w_l(y)]\Big) \;=\; 0 \end{aligned}$$

where $\alpha_{l-1\,*y}(w_l(y)) = \beta_{l-1\,*y}(w_l(y)) = 0$ because α_{l-1} and β_{l-1} are constant along the trajectories of w_l and $[w^1_{l-1}(y), w_l(y)] = [w^2_{l-1}(y), w_l(y)] = 0$ because $(w^1_1, \dots, w^1_l)$ and $(w^2_1, \dots, w^2_l)$ are generating frame fields respectively of $\mathcal{F}_1$ and $\mathcal{F}_2$ with $w^1_l = w^2_l = w_l$.

It is convenient to explain explicitly the next inductive step.

For $i = l-2$ we consider a partition of unity $\mathcal{P}_{l-2} = \{\alpha_{l-2}(y), \beta_{l-2}(y)\}$ subordinate to $\{U_1, U_2\}$ which is constant this time along each trajectory of w_l and along each trajectory of w_{l-1} (so constant along the whole of each integral surface generated by the 2-frame (w_{l-1}, w_l)). Then define

$$w_{l-2}(y) = \alpha_{l-2}(y) w^1_{l-2}(y) + \beta_{l-2}(y) w^2_{l-2}(y\,).$$

As above we find that

$$[w_{l-2}(y), w_l(y)] = [w_{l-2}(y), w_{l-1}(y)] = 0\,.$$

For an arbitrary $i < l-1$, after having constructed the vector fields $w_l, \dots, w_i$ whose Lie brackets are zero, then the inductive $(i-1)$-step can be obtained by considering a partition of unity $\mathcal{P}_{i-1} = \{\alpha_{i-1}(y), \beta_{i-1}(y)\}$ of $\mathbb{R}^k$ subordinate to $\{U_1, U_2\}$ which is constant along the trajectories of all vector fields $w_l, \dots, w_i$, so constant along each integral manifold generated by $(w_i, \dots, w_{l-1}, w_l)$, and defining

$$w_{i-1} \;=\; \alpha_{i-1} w^1_{i-1} \;+\; \beta_{i-1} w^2_{i-1}\,.$$

In this way the frame field $(w_1, \dots, w_l)$ obtained at the end of the induction will satisfy

$$[w_i, w_j] \;=\; 0 \quad , \quad \forall\, i,j = 1, \dots, l$$

and :

$$\begin{cases} p_{|Y*y}(w_i(y)) = \alpha_i(y) p_{|Y*y}(w^1_i(y)) + \beta_i(y) p_{|Y*y}(w^2_i(y)) = \alpha_i(y)\cdot E_i + \beta_i(y)\cdot E_i = E_i \\ \rho_{XY*y}(w_i(y)) = \alpha_i(y)\rho_{XY*y}(w^1_i(y)) + \beta_i(y)\rho_{XY*y}(w^2_i(y)) = \alpha_{l-1}(y)\cdot 0 + \beta_{l-1}(y)\cdot 0 = 0\,. \end{cases}$$

Thus $(w_1, \dots, w_l)$ generates a nice foliation $\mathcal{F}$.

If moreover $\mathcal{F}_1$ and $\mathcal{F}_2$ are both δ-close to an l-distribution $\mathcal{D}$ of Y then, with the canonical liftings $(v_1, \dots, v_l)$ of $(E_1, \dots, E_l)$ on $\mathcal{D}$ (see Definition 5), one has :

$$||w^1_i(y) - v_i(y)|| \le \delta \quad \text{ and } \quad ||w^2_i(y) - v_i(y)|| \le \delta\,, \qquad \forall\, i = 1, \dots, l$$

hence for every $i = 1, \dots, l$ one also finds :

$$||w_i(y) - v_i(y)|| = \alpha_i(y)||w^1_i(y) - v_i(y)|| + \beta_i(y)||w^2_{i-1} - v_i(y)|| \le \big(\alpha_i(y) + \beta_i(y)\big)\cdot\delta \;=\; 1\cdot\delta \;=\; \delta$$

so that the foliation $\mathcal{F} := \mathcal{F}_1 \vee \mathcal{F}_2$ generated by $(w_1, \dots, w_l)$ is δ-close to $\mathcal{D}$ too. □

With essentially the same proof as in Theorem 2 one has :

Corollary 1. *Let $h_i : \mathbb{R}^k \to \mathbb{R}^k$ be the linear diffeomorphism which permutes x_1 with x_i and fixes all other coordinates and consider the open covering of $T_{XY}(\epsilon)$ defined by :*

$$U^i_1 \;:=\; T_{XY}(\epsilon) \cap h_i(]-\infty, a[\times\mathbb{R}^{n-1}) \quad \textit{and} \quad U^i_2 := T_{XY}(\epsilon) \cap h_i(]b, +\infty[\times\mathbb{R}^{n-1})\,.$$

If $i \in \{2, \ldots, l-1\}$ and $\mathcal{F}_1^i$ and $\mathcal{F}_2^i$ are nice foliations of $T_{XY}(\epsilon)$ then Theorem 2 holds again replacing U_1 and U_2 by U_1^i and U_2^i so as to glue together $\mathcal{F}_1^i$ and $\mathcal{F}_2^i$. $\square$

Remark 3 On the contrary, for $i = l$, Theorem 2 cannot be used directly to glue together two nice foliations $\mathcal{F}_1^l$ and $\mathcal{F}_2^l$ of U_1^l and U_2^l, because if $w_l^1 = w_l^2$, then a partition of unity subordinate to $\{U_1^l, U_2^l\}$ cannot be taken constant along the trajectories of $w_l^1 = w_l^2$.

The techniques of Theorem 2 and Corollary 1 will be used in various steps of our proof of the smooth version of the Whitney fibering conjecture (Theorem 3) while the difficulty explained in Remark 3 above will appear in step 3 of the proof.

5. The smooth version of the Whitney fibering conjecture.

In this section we prove our main result (Theorem 3), which gives a positive answer to a smooth version of the Whitney fibering conjecture on X for a stratification $\mathcal{X} = (A, \Sigma)$ in which $depth_\Sigma(X) = 1$ and which is Bekka (c)-regular **[Be]** on X.

Theorem 3. *Let $\mathcal{X} = (A, \Sigma)$ a smooth stratified Bekka (c)-regular subset of $\mathbb{R}^n$.*

Then for every stratum X of $depth_\Sigma(X) = 1$ and for every $x_0 \in X$ and every stratum $Y > X$ there exists a neighbourhood W of x_0 in $X \cup Y$ and a foliation $\mathcal{H} = \{M'_y\}_y$ of W whose leaves M'_y are smooth manifolds of dimension $l = \dim X$ diffeomorphic to $X \cap W$ and such that :

$$\lim_{y \to x} T_y M'_y \;=\; T_x M'_x \;=\; T_x X\,, \qquad \textit{for every} \;\; x \in X \cap W\,.$$

Remark 4. In Theorem 3 we study a (c)-regular stratification with smooth (C^∞) strata and obtain a foliation which is C^∞ off X. If the stratification has C^1 strata there is a C^1 diffeomorphism making all strata C^∞ **[Tr]**$_2$ so we can apply the C^∞ result and then by pullback obtain a foliation with C^1 leaves.

Before proving Theorem 3 in §5.1 we describe local regularity of the stratified topological triviality map H_{x_0} and some of its important properties when H_{x_0} is obtained by integrating *continuous canonical* lifted frame fields **[MT]**$_{2,3,4}$. This brings us in §5.2 to a finer analysis of some new properties of the frame fields tangent to the horizontal leaves $\mathcal{H}_{x_0}$ defined by this topological trivialization.

We will use below statements and notations introduced in section 3.

5.1. Local topological triviality obtained from continuous lifted frame fields.

Let $\mathcal{X}$ be a (c)-regular stratification in a Euclidian space $\mathbb{R}^k$, X a stratum of $\mathcal{X}$ of dimension l, $x_0 \in X$ and $U_{x_0} \cong \mathbb{R}^l$ a neighbourhood of x_0 in X as in Theorem 1.

In a local analysis we can suppose $x_0 = 0 \in \mathbb{R}^k$, $U_{x_0} = \mathbb{R}^l \times \{0^m\}$ and $\pi_X : T_X \to \mathbb{R}^l \times \{0^m\}$ is the canonical projection such that the topological trivialization "with origin $x_0 = 0$" of the projection π_X can be written as

$$\begin{array}{rcccl} H \;=\; H_{x_0} & : & \mathbb{R}^l \times \pi_X^{-1}(x_0) & \longrightarrow & \pi_X^{-1}(U_{x_0}) \subseteq \mathbb{R}^n \\ & & (t_1, \ldots, t_l, z_0) & \longmapsto & \phi_l(t_l, \ldots, \phi_1(t_1, z_0) \ldots) \end{array}$$

where $\forall i \leq l$, ϕ_i is the flow of the lifted vector field $v_i(y)$, and thanks to (c)-regularity (**[Be]** **[Pl]**), we can choose each $v_i(y)$ to be the *continuous* lifting of the standard vector fields E_i of $X = \mathbb{R}^l \times \{0^m\}$, in a canonical distribution $\mathcal{D}_X = \{\mathcal{D}_X(y)\}_{y \in \pi_X^{-1}(U_{x_0})}$ induced from X on the strata $Y > X$ **[MT]**$_{2,3,4}$. We also will identify $T_{XY}(\epsilon)$ with Y.

Remark 5. Although the canonical distribution $\mathcal{D}_X$, its spanning canonical lifted vector fields $v_1, \ldots, v_l$ and their flows $\phi_1, \ldots, \phi_l$ do not depend on the "starting point" x_0, in contrast the trivialization H_{x_0}, defined by a fixed and *a priori non-commuting* order of composition of the flows $\phi_1, \ldots, \phi_l$, depends strongly on the induced nice foliations from a "starting point" x_0. In fact the non-commutativity of the flows $\phi_1, \ldots, \phi_l$ is the crucial point of our problem : if $\mathcal{D}_X$ is involutive, then the (a)-regularity of a local horizontal foliation $\mathcal{H}_{x_0}$ holds ($[\mathbf{MT}]_4$ and $[\mathbf{Mu}]_1$ Chap. II §5) so $\mathcal{H}_{x_0}$ satisfies the conclusions of the smooth version of the Whitney fibering conjecture (see section 2).

In particular if $x = (\tau_1, \ldots, \tau_l)$ and $z \in \pi_X^{-1}(x)$ is the image $z = H_{x_0}(\tau_1, \ldots, \tau_l, z_0)$ with $z_0 \in \pi_{XY}^{-1}(x_0)$ then :

$$y \;=\; H_x(t_1, \ldots, t_l, z) \;=\; \phi_l(t_l, \ldots, \phi_1(t_1, z)..) \;=\; \phi_l(t_l, \ldots, \phi_1(t_1,\; (\phi_l(\tau_l, \ldots, \phi_1(\tau_1, z_0)..)..)$$

is a priori different from the image (obtained by commuting the flows ϕ_i) :

$$\phi_l(t_l + \tau_l, \ldots, \phi_1(t_1 + \tau_1, z_0)) \;=\; H_{x_0}(t_1 + \tau_1, \ldots, t_l + \tau_l, z_0)\,.$$

Let $Y > X$.

The stratified homeomorphism H (a C^∞-diffeomorphism on each stratum) induces a "horizontal" foliation of dimension l

$$\mathcal{H}_{x_0} := \Big\{ M_{z_0} = H(\mathbb{R}^l \times \{z_0\}) \Big\}_{z_0 \in \pi_{XY}^{-1}(x_0)}$$

of the submanifold $\pi_{XY}^{-1}(U_{x_0})$ of Y.

For every $y \in Y$ let us denote by M_y the leaf of $\mathcal{H}_{x_0}$ containing y, so that $M_y = M_{y_0}$ when $y = H(t_1, \ldots, t_l, y_0)$ and $y_0 \in \pi_{XY}^{-1}(x_0)$.

We will see in Proposition 3, that writing $\forall\, i = 1, \ldots, l$, $w_i(y) := H_{*(t_1, \ldots, t_l, y_0)}(E_i)$, the frame field $(w_1, \ldots, w_l)$ is the unique (π, ρ)-controlled lifting on the foliation $\mathcal{H}_{x_0}$ (not necessarily continuous) of the frame field $(E_1, \ldots, E_l)$ of X generating $\mathcal{H}_{x_0}$ (see $[\mathbf{MT}]_4$, $[\mathbf{Mu}]_1$ Chap. II, §5).

Now H being smooth on Y, the $w_1, \ldots, w_l$ are smooth too on Y, but these vector fields are not necessarily continuous on X, i.e. <u>we do not know</u> whether $\lim_{y \to x} w_i(y) = E_i$ for $x \in X$!

This means that by using the canonical continuous liftings $v_1, \ldots, v_l$ on $\mathcal{D}_{XY}$, the foliation H_{x_0} induced by the topological trivialization of Thom-Mather, does not in general give a positive answer to the smooth version of the Whitney fibering conjecture (du Plessis and Trotman gave an explicit counterexample in 1994).

5.2. Some useful properties of the w_i and of their flows.

We explain below a property of the vector fields v_i (and w_i) and of their flows ϕ_i (and ψ_i), important in the proof of the smooth Whitney fibering conjecture.

The vector fields $w_1, \ldots, w_l$ satisfy obviously :

$$w_i(z_0) = H_{*(0, \ldots, 0, z_0)}(E_i) = v_i(z_0), \qquad \forall\, z_0 \in \pi_{XY}^{-1}(x_0) \quad \text{and} \quad \forall\, i = 1, \ldots, l.$$

That is for every $i \leq l$, w_i coincides on the fiber $\pi_{XY}^{-1}(x_0)$ with the continuous lifting v_i $[\mathbf{MT}]_4$ which satisfies $\lim_{y \to x \in X} v_i(y) = E_i$, for every $i = 1, \ldots, l$ (but again this does not imply that $\lim_{y \to x} w_i(y) = E_i$ for $x \in X$!).

Now if $y = H(t_1, \dots, t_l, y_0)$, $M_y = H(\mathbb{R}^l \times \{y_0\})$ is a leaf of the foliation $\mathcal{H}_{x_0}$ and

$$T_y M_y = H_{*(t_1,\dots,t_l,y_0)}(\mathbb{R}^l \times \{0\}) = [w_1(y), \dots, w_l(y)] .$$

On the other hand $H_{|Y}$ being C^∞, for each $z_0 \in \pi_{XY}^{-1}(x_0)$ we have that :

$$(L_{z_0}) : \qquad \lim_{y \to z_0} T_y M_y \;=\; T_{z_0} M_{z_0} \;=\; [w_1(z_0), \dots, w_l(z_0)] \;=\; [v_1(z_0), \dots, v_l(z_0)] .$$

Lemma 1. *For every $y_0 \in \pi_{XY}^{-1}(x_0)$, denoting $Q_0(\delta) =]-\delta, \delta[^l$, the family*

$$\Big\{ H(Q_0(\delta) \times J_{y_0}) \quad \Big| \quad \delta \in]0,1[\,, \quad J_{y_0} \text{ a neighbourhood of } y_0 \text{ in } \pi_{XY}^{-1}(x_0) \Big\}$$

is a fundamental system of neighbourhoods of y_0 in Y.

Proof. Exercise. □

From (L_{z_0}) for every $\epsilon > 0$, there is a relatively compact open neighbourhood V_{z_0} of z_0 in $\pi_{XY}^{-1}(U_{x_0})$ such that

$$(1) \quad : \qquad ||w_i(y) - v_i(y)|| < \epsilon \qquad \forall\, i = 1, \dots, l \;\text{ and }\; \forall\, y \in V_{z_0} .$$

By Lemma 1, we can take $V_{z_0} = H_{x_0}(Q_0(\delta_{z_0}) \times J_{z_0})$ so that

$$I_{x_0} \;:=\; pr_{\mathbb{R}^l \times 0^m}(V_{z_0}) \;=\; x_0 \,+\, Q_0(\delta_{z_0}) \;=\,]-\delta_{z_0},\, +\delta_{z_0}[^l$$

is a relatively compact open neighbourhood of $x_0 = 0$ in $\widetilde{U} := [-1,1]^l \times \{0\}^m$ which is a *cube* of $\mathbb{R}^l \times \{0^m\}$ centered in x_0.

Definition 7. We will refer to the property (1) by saying that :

The foliation $\mathcal{H}_{x_0} = \{H_{x_0}(Q_0(\delta_{z_0}) \times \{z'\})\}_{z' \in J_{z_0}}$ on V_{z_0} is ϵ-close to the canonical distribution $\mathcal{D}_X = \{\mathcal{D}_{XY}(y)\}_{y \in Y}$.

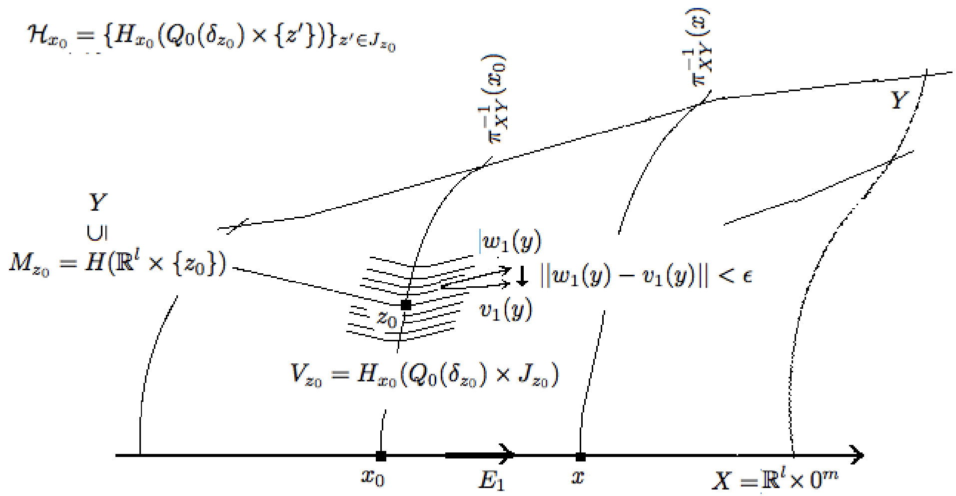

Figure 3

To analyse the difference between $v_i(y)$ and $w_i(y) = H_{*(t_1,\dots,t_l,y_0)}(E_i)$ we introduce the following notations.

Notations. With every $y = H(t_1, \ldots, t_l, y_0) \in Y$ we associate the chain $y_0 \cdots y_i \cdots y_l = y$ defined starting from y_0 on the leaf $M_y = H(y_0 \times \mathbb{R}^l)$ of the foliation $\mathcal{H}_{x_0}$ as follows :

$$\begin{cases} y_0 & = H(0^l, y_0) \\ y_1 & = H(t_1, 0^{l-1}, y_0) = \phi_1(t_1, y_0) \\ y_2 & = H(t_1, t_2, 0^{l-2}, y_0) = \phi_2(t_2, \phi_1(t_1, y_0)) \\ \ldots & \quad \ldots\ldots\ldots\ldots\ldots \\ y_i & = H(t_1, \ldots, t_i, 0^{l-i}, y_0) = \phi_i(t_i, \ldots, \phi_1(t_1, y_0) \ldots) \\ \ldots & \quad \ldots\ldots\ldots\ldots\ldots \\ y_l & = H(t_1, \ldots, t_l, y_0) = \phi_l(t_l, \ldots, \phi_i(t_i, \ldots, \phi_1(t_1, y_0) \ldots) \ ; \end{cases}$$

so that :

$$y_1 = \phi_1(t_1, y_0), \quad y_2 = \phi_2(t_2, y_1), \quad \ldots \quad y_i = \phi_i(t_i, y_{i-1}), \quad \ldots \quad y_l = \phi_l(t_l, y_{l-1}) = y \ .$$

In the proposition below, $\forall \tau \in \mathbb{R}$, we let $\phi_i^\tau : Y \to Y$ be the diffeomorphism of Y defined by $\phi_i^\tau(y) = \phi_i(\tau, y)$.

Proposition 2. *For every* $y = H(t_1, \ldots, t_l, y_0) \in Y$*,*

$$w_i(y) = \phi^{t_l}_{l\ *y_{l-1}} \circ \cdots \circ \phi^{t_{i+1}}_{i+1\ *y_i}\big(v_i(y_i)\big) \qquad , \qquad \forall\, i = 1, \ldots, l-1 \ .$$

Proof. As $y = H(t_1, \ldots, t_l, y_0)$ it follows that :

$$\begin{aligned} w_i(y) \ &:= \ H_{*(t_1,\ldots,t_l,y_0)}(E_i) \ = \ \frac{\partial}{\partial \tau_i} H(\tau_1, \ldots, \tau_l, y_0)\Big|_{(\tau_1,\ldots,\tau_l)=(t_1,\ldots,t_l)} \\ &= \ \frac{\partial}{\partial \tau_i}\Big|_{(\tau_1,\ldots,\tau_l)=(t_1,\ldots,t_l)} \phi_l(\tau_l, \ldots, \phi_i(\tau_i, \ldots, \phi_1(\tau_1, y_0) \ldots) \\ &= \ \frac{\partial}{\partial \tau_i}\Big|_{\tau_i = t_i} \phi_l^{t_l} \circ \cdots \circ \phi_{i+1}^{t_{i+1}} \circ \phi_i^{\tau_i}(y_{i-1}) \\ &= \ \phi^{t_l}_{l\ *y_{l-1}} \circ \cdots \circ \phi^{t_{i+1}}_{i+1\ *y_i}\left(\frac{\partial}{\partial \tau_i}\Big|_{\tau_i = t_i} \phi_i(\tau_i, y_{i-1}) \right) \\ &= \ \phi^{t_l}_{l\ *y_{l-1}} \circ \cdots \circ \phi^{t_{i+1}}_{i+1\ *y_i}\Big(v_i\big(\phi_i(t_i, y_{i-1})\big)\Big) \\ &= \ \phi^{t_l}_{l\ *y_{l-1}} \circ \cdots \circ \phi^{t_{i+1}}_{i+1\ *y_i}\big(v_i(y_i)\big) \ . \quad \square \end{aligned}$$

For every $y \in Y$ and leaf M_y, denote ${\pi_{XY}}^{-1}(x_0) = S^0 \subseteq S^1 \subseteq \cdots \subseteq S^l = M_y$ the chain of "coordinate subspaces" S^i of Y containing all points of the type $y = y_i$:

$$S^i \ := \ H\big(\mathbb{R}^i \times 0^{l-i} \times {\pi_{XY}}^{-1}(x_0)\big) =$$

$$= \ \Big\{ y = H(t_1, \ldots, t_i, 0^{l-i}, y_0) \ \big| \ y_0 \in {\pi_{XY}}^{-1}(x_0) \ , \ t_1, \ldots, t_i \in \mathbb{R} \Big\} .$$

Then every S^i is a submanifold of dimension $i + (k-l)$ of Y, where $k = \dim Y$ $(k - l = \dim {\pi_{XY}}^{-1}(x_0))$, and one has :

Corollary 2. *For every $i = 1, \dots, l$, the vector field $w_i(y)$ coincides with the lifting $v_i(y)$ in the canonical distribution $\mathcal{D}_{XY}(y)$ on all points of the submanifold S^i :*

$$w_i(y) = v_i(y) \quad , \qquad \forall\, y \in S^i \ .$$

In particular $\forall\, i = 1, \dots, l$, the flow ψ_i of w_i coincides with the flow ϕ_i of v_i on $S^i \times \mathbb{R}$.

Proof. If a point $y = H(t_1, \dots, t_l, y_0)$ coincides with the corresponding y_i then necessarily $t_{i+1} = \dots = t_l = 0$ and also $y = y_l = y_{l-1} = \dots = y_i$.

For every $j = i+1, \dots, l$, since $t_j = 0$ the flows satisfy $\phi_j^{t_j} = \phi_j^0 = 1_Y$ and $\phi_{j*y_j}^{t_j} = 1_{Y*y_j} = 1_{T_{y_j}Y}$ and so by the previous proposition one finds :

$$w_i(y) \;=\; \phi^{t_l}_{l\ *y_{l-1}} \;\circ\; \cdots \;\circ\; \phi^{t_{i+1}}_{i+1\ *y_i}\big(v_i(y_i)\big) \;=\; v_i(y_i) \;=\; v_i(y)\ . \qquad \square$$

Corollary 2 allows us to better estimate the difference $u_i(y) := v_i(y) - w_i(y)$: it increases for i decreasing, being zero for $i = l$ and maximal when $i = 1$. This is a consequence of the nature of the definition of the trivialization H,

$$H(t_1, \dots, t_l, y_0) = \phi_l(t_l, \dots, \phi_i(t_i, \dots, \phi_1(t_1, y_0) \dots)$$

because of which any vector $w_i(y)$ whose index i is more to the left of the formula relating $w_i(y_i)$ to $v_i(y_i)$ occurs in the " perturbation " of the extra differential $\phi^{t_i+1}_{i+1\ *y_i}$ compared with the previous pair $w_{i+1}(y_{i+1})$, $v_{i+1}(y_{i+1})$.

Thus since $S^l = Y$ and $S^0 = \pi_{XY}^{-1}(x_0)$, the vector fields $w_1(y), \dots, w_l(y)$ satisfy :

$$\begin{cases} w_l(y) = v_l(y) & \text{on} \quad S^l \ = Y \\ \dots \ = \ \dots & \\ w_i(y) = v_i(y) & \text{on} \quad S^i \ = \ H\big(\mathbb{R}^i \times 0^{l-i} \times {\pi_{XY}}^{-1}(x_0)\big) \\ \dots \ = \ \dots & \\ w_1(y) \ = \ v_1(y) & \text{on} \quad S^1 \ = \ H\big(\mathbb{R}^1 \times 0^{l-1} \times \pi_{XY}^{-1}(x_0)\big)\,. \end{cases}$$

This explains why the order of the index $i = 1, \dots, l$ that we have chosen to define the topological trivialization H is significant!

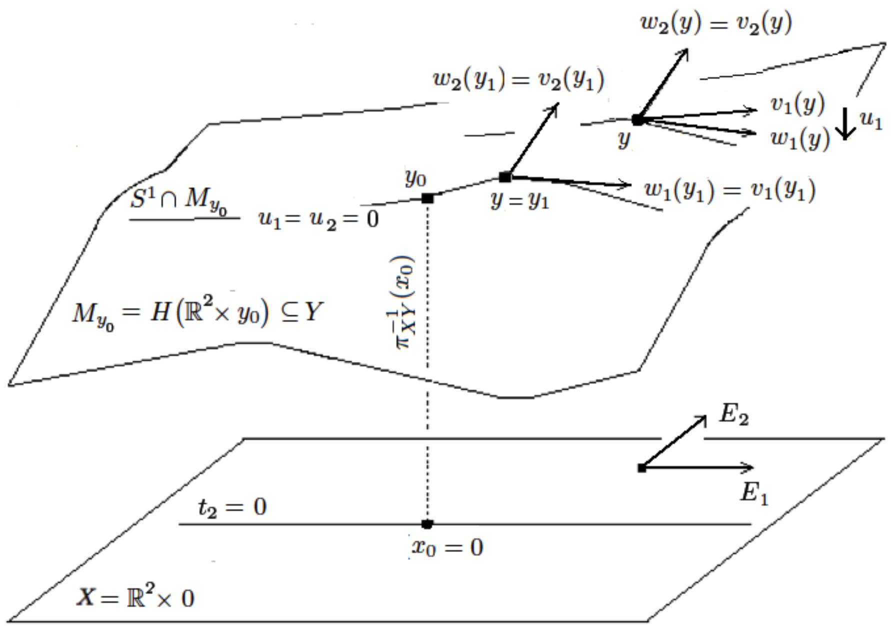

Figure 4

The vector fields $\{w_i(y)\}_{i=1,\dots,l}$ are characterised by the following property :

Proposition 3. *Every vector field $w_i(y)$ is the unique (π,ρ)-controlled lifting of the standard vector field E_i of X tangent to the leaves of the foliation $\mathcal{H}_{x_0}$.*

Proof. See **[MT]**$_4$, §5.1 Lemma 3. □

5.3. Proof of the smooth version of the Whitney fibering conjecture for (c)-regular stratifications of depth 1.

Let $\mathcal{X} = (A,\Sigma)$ be a (c)-regular stratification, X an l-stratum of $\mathcal{X}$ and $x_0 \in X$.

Since by hypothesis $depth_\Sigma(X) = 1$, a small neighbourhood of x_0 in A meets only finitely many strata $\{Y_i^{k_i} > X\}_{i=1}^r$ **[Ma]** which are of dimension $\dim Y_i^{k_i} = k_i \geq l+1 > \dim X$, and the closure of these strata intersect (near X) only X. Therefore it will be sufficient to prove Theorem 3 for only one stratum Y. So we prove:

Theorem 4. *Suppose that $X \cup Y$ is a smooth stratified Bekka (c)-regular closed subset of $\mathbb{R}^n$ having only two smooth strata $X < Y$. Then $X \cup Y$ satisfies the smooth version of the Whitney fibering conjecture. I.e. for every $x_0 \in X$ there exists a neighbourhood W of x_0 in $X \cup Y$ and a foliation $\mathcal{H} = \{M'_y\}_y$ of W whose leaves M'_y are smooth manifolds diffeomorphic to $X \cap W$ and such that :*

$$\lim_{y \to x} T_y M'_y \;=\; T_x M'_x \;=\; T_x X\,, \qquad \textit{for every} \;\; x \in X \cap W\,.$$

Proof. Let $l = \dim X$, $k = \dim Y$ and let x_0 be a point of X and $U := U_{x_0}$ an open neighbourhood of x_0 in X diffeomorphic to $\mathbb{R}^l$.

When $l = 1$, (c)-regularity implying the existence of controlled <u>continuous</u> lifting of vector fields on X, is enough to ensure the existence of the foliation **[Be]**.

So we may assume $l \geq 2$.

When $k = l+1$, the level hypersurfaces of ρ_{XY} intersect Y in leaves of an appropriate foliation, again by (c)-regularity **[MT]**$_4$. Thus we will assume $k \geq l+2 \geq 4$.

The problem being local, we can suppose that $x_0 = 0^n$, $X = \mathbb{R}^l \times \{0^m\}$ $(m = n-l)$, $Y = T_{XY} = \pi_{XY}^{-1}(U)$ where the projection $\pi_{XY} : T_{XY} = T_X \cap Y \to X$ is the restriction $pr_{1|Y} : T_{XY} \to X$ of the first projection onto $\mathbb{R}^l \times \{0\} = X$; in particular $\pi_{XY}^{-1}(x_0) \subseteq \{0^l\} \times \mathbb{R}^m \subseteq \mathbb{R}^n$.

Recall the standard basis $\{E_i\}_{i=1}^l$ of $\mathbb{R}^l \times \{0^m\}$ and the topological trivialization "of origin x_0" of the projection π_{XY}:

$$\begin{array}{rcccl} H \;=H_{x_0} & : & \mathbb{R}^l \times \pi_{XY}^{-1}(x_0) & \longrightarrow & Y = T_{XY} \subseteq \mathbb{R}^n \\ & & (t_1,\dots,t_l,z_0) & \longmapsto & \phi_l(t_l,\dots,\phi_1(t_1,z_0)\dots) \end{array}$$

where $\forall i \leq l$, ϕ_i is the flow of the vector field v_i which is the <u>*continuous*</u> lifting of E_i in a canonical distribution $\mathcal{D}_X = \{\mathcal{D}_{XY}(y)\}_{y\in Y}$ induced from X on Y **[MT]**$_{2,3}$.

As $X \cup Y$ is (c)-regular, there exists $\epsilon > 0$ such that the map $(\pi_{XY},\rho_{XY}) : T_{XY}(\epsilon) \to \mathbb{R}^l \times [0,\infty[$ is a proper submersion and, making possibly a change of scale, we may suppose $\epsilon = 1$. Then if we consider the compact neighbourhood $\widetilde{U} = [-1,1]^l \times \{0\}^m$ of $x_0 = 0^k$ in X, its preimage $W := \pi_X^{-1}(\widetilde{U})$ via the projection $\pi_X : T_X(1) \to X$ is a compact subset of $T_X(1) = X \cup Y$.

From now on we will consider always points of X lying in U and points of Y lying in $\pi_{XY}^{-1}(U)$ so we identify X with U and $Y = T_{XY}$ with $\pi_{XY}^{-1}(U)$.

We apply the arguments described in sections 5.1 and 5.2 for x_0 and for $y_0 \in \pi_{XY}^{-1}(x_0)$ to each point $x \in \widetilde{U}$ and each $z \in \pi_{XY}^{-1}(x)$. Every $x \in \widetilde{U}$ will be thus "the origin" of a new topological trivialization H_x obtained using the same continuous lifted vector fields $v_1, \dots, v_l$ by composing their flows in the same order, but taking x as origin. This will define for every $x \in \widetilde{U}$ a foliation

$$\mathcal{H}_x = \Big\{ M_z^x = H_x(\mathbb{R}^l \times \{z\}) \Big\}_{z \in \pi_{XY}^{-1}(x)}$$

and a (π, ρ)-controlled frame field $(w_1^x, \dots, w_l^x)$ generating the foliation $\mathcal{H}_x$, such that $\forall\, x \in \widetilde{U}$ and $\forall\, z \in \pi_{XY}^{-1}(x)$:

$$(L_z): \qquad \lim_{y \to z} T_y M_y^x \;=\; \lim_{y \to z} T_y M_z^x \;=\; [w_1^x(z), \dots, w_l^x(z)] = [v_1(z), \dots, v_l(z)] = \mathcal{D}_{XY}(z)\,,$$

with $\mathcal{D}_{XY}(y)$ which tends continuously to $[E_1, \dots, E_l] = \mathbb{R}^l \times \{0^m\} = T_x X$, as $y \to x$, by (c)-regularity.

From now on we will suppose that $\dim X = l = 2$. Later in the proof we will treat the general case.

With such a hypothesis $X = \mathbb{R}^2 \times \{0^m\}$ and by the results of section 5.1 and 5.2 we have the (π, ρ)-controlled continuous lifted frame field (v_1, v_2) on $\mathcal{D}_{XY}$, a frame field $(w_1^{x_0}, w_2^{x_0})$ tangent to the foliation $\mathcal{H}_{x_0}$ and for each $x \in \widetilde{U}$ a frame field (w_1^x, w_2^x) tangent to the foliation $\mathcal{H}_x$ such that :

1) for every $x \in X$: $\quad w_2^{x_0} = v_2 = w_2^x$;

2) for every $y = H_{x_0}(t_1, t_2, y_0)$, with $y_0 \in \pi_{XY}^{-1}(x_0)$, by setting $y_1 = \phi_1(t_1, y_0)$ we have:

$$w_1^{x_0}(y) \;=\; \phi_{2*y_1}^{t_2}(v_1(y_1))\,,$$

and similarly for every $z = H_x(t_1, t_2, z_0)$, with $z_0 \in \pi_{XY}^{-1}(x)$, by setting $z_1 = \phi_1(t_1, z_0)$ we have :

$$w_1^x(z) \;=\; \phi_{2*z_1}^{t_2}(v_1(z_1))\,.$$

3) by (L_z) applied to each $z \in \pi_{XY}^{-1}(x)$, for every $z \in W$ and for every $\epsilon > 0$, there is a relatively compact open neighbourhood V_z of z in W such that

$$||w_1^x(y) - v_1(y)|| < \epsilon \qquad \text{for every} \quad y \in V_z\,.$$

By Lemma 1, we can choose every V_z to be of the type

$$V_z = H_x(Q_0(\delta_z) \times J_z) \qquad \text{where} \qquad x = \pi_{XY}(z) = (\tau_1, \tau_2)\,,$$

(see Lemma 1 in §5.2 for the definitions of $Q_0(\delta_z)$ and J_z) where :

$$I_x \;:=\; pr_{\mathbb{R}^2 \times 0^m}(V_z) \;=\; x \,+\, Q_0(\delta_z) \;=\; (\tau_1, \tau_2) \,+\,]-\delta_z,\, +\delta_z[^2$$

is a relatively compact open neighbourhood of x in $\widetilde{U} = [-1, 1]^2 \times \{0\}^m$ and a *square* of $\mathbb{R}^2 \times \{0^m\}$ centered in x with edges of size $2\delta_z$ depending on z, and where J_z is a relatively compact open neighbourhood of z in $\pi_{XY}^{-1}(x)$.

In this way, the foliation (called again) $\mathcal{H}_x = \left\{H_x(Q_0(\delta_z) \times \{z'\})\right\}_{z' \in J_z}$ on V_z will be ϵ-close to the canonical distribution $\mathcal{D}_X = \{\mathcal{D}_{XY}(y)\}_{y \in Y}$.

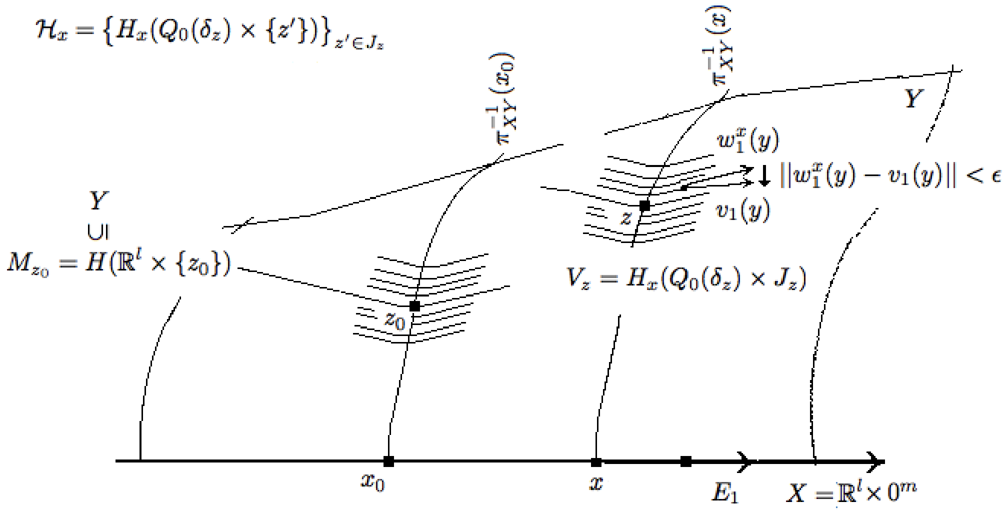

Figure 5

For every $n \in \mathbb{N}^*$ let

$$F_n \ := \ (\pi_{XY}, \rho_{XY})^{-1}(\{x_0\} \times [\tfrac{1}{n+2}, \tfrac{1}{n}]) \ = \ \pi_{XY}^{-1}(x_0) \cap \rho_{XY}^{-1}([\tfrac{1}{n+2}, \tfrac{1}{n}])$$

and let A_n be the compact cylindrical set of $\pi_{XY}^{-1}([-1,1]^2)$:

$$A_n = (\pi_{XY}, \rho_{XY})^{-1}([-1,1]^2 \times [\tfrac{1}{n+2}, \tfrac{1}{n}]) = \pi_{XY}^{-1}([-1,1]^2) \cap \rho_{XY}^{-1}([\tfrac{1}{n+2}, \tfrac{1}{n}])$$

so that :

$$A_n = H_{x_0}([-1,1]^2 \times F_n) \qquad \text{and} \qquad F_n = \pi_{XY}^{-1}(x_0) \cap A_n \,.$$

We now make more precise the geometric properties that we require of the neighbourhoods I_x of $x = \pi_{XY}(z)$ in U and the neighbourhoods J_z of z in $\pi_{XY}^{-1}(x)$.

Consider for every $n \in \mathbb{N}^*$ and every $z \in A_n \subseteq Y$, the topological trivialization $H_x : \mathbb{R}^2 \times \pi_{XY}^{-1}(x) \to Y$, with $x = \pi_{XY}(z)$.

Fix $\epsilon := \epsilon_n = \frac{1}{n}$ with the previous property 3), $\|w_1^x(y) - v_1(y)\| < \frac{1}{n}$ for $z \in A_n$.

Since the set

$$\mathcal{S}_n \ := \ \left\{ \ V_z = H_x(Q_0(\delta_z) \times J_z) \quad \middle| \quad z \in A_n \ , \ \ \delta_z \in]0,1[\ \right\}$$

is an open covering of A_n, and A_n is compact, there exists a finite subset of points of A_n, $P_n := \{z_1, \ldots, z_{q_n}\}$ such that the open finite subcovering

$$\mathcal{C}_n \ := \ \left\{ \ V_{z_j} = H_{x_j}(Q_0(\delta_{z_j}) \times J_{z_j}) \ \middle| \ z_j \in P_n \ \right\}, \quad \text{where} \quad x_j = \pi_{XY}(z_j)\,,$$

covers A_n.

Recall that since every local trivialization H_{x_j} is π-controlled, $\pi_{XY*y}(w_i^{x_j}(y)) = E_i$.

Moreover, since H_{x_j} is ρ-controlled, $\rho_{XY*y}(w_i^{x_j}(y)) = 0$, so that the vector fields $w_i^{x_j}(y)$ have only components (apart from the E_i) along the tangent space to a link of the fiber $\pi_{XY}^{-1}(\pi_{XY}(y))$:

$$L(y) \ := \ (\pi_{XY}, \rho_{XY})^{-1}((\pi_{XY}(y), \rho_{XY}(y))) \ = \ \pi_{XY}^{-1}(\pi_{XY}(y)) \cap \rho_{XY}^{-1}(\rho_{XY}(y))$$

which is a compact $(k-3)$-submanifold of $\pi_{XY}^{-1}(\pi_{XY}(y))$.

We will prove that this finite open covering $\mathcal{C}_n$ of A_n by nicely foliated $\frac{1}{n}$-close to $\mathcal{D}_X$, open sets $\{V_{z_j}\}_{j\le q_n}$, can be used to define a more convenient foliation defined on the whole annulus A_n, which is again $\frac{1}{n}$-close to $\mathcal{D}_{XY}$.

Let η_n^0 be the Lebesgue number of the open covering $\mathcal{C}_n = \{ \ V_{z_j} \mid j = 1, \ldots, q_n\}$, so that every subset of A_n of diameter $< \eta_n^0$ is contained in at least one of the sets V_{z_j}, for some $j = 1, \ldots, q_n$.

Similarly, H being π-controlled, the family $\{I_{z_j} := x_j + Q_0(\delta_{z_j})\}_{j\le q_n}$ is an open covering of $[-1,1]^2$, so if we denote by η_n^1 its Lebesgue number, then every subset of $[-1,1]^2$ of diameter $< \eta_n^1$ is contained in at least one of the cubes $I_{z_j} = x_j + Q_0(\delta_{z_j})$.

Moreover, for every $x_j = (t_1^j, t_2^j,) \in [-1,1]^2$, since we have a restriction homeomorphism $H_| : (\pi_{XY}^{-1}(x_0), x_0) \to (\pi_{XY}^{-1}(x_j), x_j)$, every open set J_{z_j} of the fiber $\pi_{XY}^{-1}(x_j)$ determines via H^{-1} an open set $J_{z_j}^0 = \phi_1(-t_1^j, \phi_2(-t_2^j, J_{z_j}))$ of the fiber $\pi_{XY}^{-1}(x_0)$, such that $H(\{x_j\} \times J_{z_j}^0) = J_{z_j}$. In this way we obtain an open covering $\{ \ J_{z_j}^0\}_{j\le q_n}$ of $F_n = \pi_{XY}^{-1}(x_0) \cap A_n$ whose Lebesgue number will be denoted by η_n^2.

We let

$$\eta_n = \min\{\eta_n^0 \, , \ \eta_n^1 \, , \ \eta_n^2 \, , \ \frac{1}{2}\} \, .$$

Now as each trivialization H_x is defined ($[\mathbf{Ma}]_{1,2}$, $[\mathbf{MT}]_4$) by the formula

$$H_x(t_1, \ldots, t_l, y_0) = \phi_l(t_l, \ldots, \phi_i(t_i, \ldots, \phi_1(t_1, y) \ldots) \, , \qquad y \in \pi_X^{-1}(x)$$

where the maps $\{\phi_i\}_{i\le l}$ are the smooth flows of the smooth canonical lifting vector fields $\{v_i\}_{i\le l}$ on Y, each H_x is smooth on Y and in particular locally Lipschitz on Y. Hence for every $j \le q_n$, the trivialization $H_{x_j|A_n}$ restricted to the compact set A_n is a globally Lipschitz map on A_n with some constant C_j (at least with respect to the geodesic arc-length metric) and hence with constant $C = max\{2, C_1, \ldots, C_{q_n}\}$.

By (c)-regularity $(\pi_X, \rho_X) : T_X(1) \to X \times [0,1]$ is a proper submersion, hence the set:

$$F_n \ = \ \pi_X^{-1}(x_0) \cap A_n \qquad \text{is a compact } (k-2)\text{-submanifold with boundary}$$

and we can choose a triangulation $T_n = T_n(x_0)$ of F_n which induces a covering by open cells of F_n :

$$\mathcal{R}_n(x_0) \ := \ \Big\{ N(\sigma) \ \big| \ \sigma \in T_n(x_0) \ , \ \ \dim \sigma = k-2 \ \Big\}$$

where $N(\sigma)$ denotes the open cellular neighbourhood of each simplex $\sigma \in T_n$ [**ST**].

For $s_n \in \mathbb{N}^*$ such that $\delta := \frac{1}{s_n} < \eta_n$ define $\Sigma = \{-s_n, \ldots, -1, 0, 1, \ldots, s_n - 1\}$ and consider the closed coverings $\big\{ \, Q_i(\delta) := Q_i = [i\delta, (i+1)\delta] \big\}_{i\in\Sigma}$ of $[-1,1]$ for which :

$$[-1,1] \quad = \quad \bigcup_{i\in\Sigma} Q_i(\delta) \ =$$

$$= \ [-s_n\delta, -(s_n-1)\delta] \ \cup \ \ldots \ \cup \ [-\delta, 0] \ \cup \ [0,\delta] \ \cup \ \ldots \ \cup \ [(s_n-1)\delta, s_n\delta] \ .$$

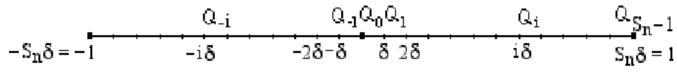

Figure 6

This covering induces the closed (paving) covering of cubes of $[-1,1]^2$:

$$\left\{ \, Q_{(i_1,i_2)}(\delta) = Q_{i_1}(\delta) \times Q_{i_2}(\delta) \right\}_{(i_1,i_2)\in\Sigma^2} \qquad \text{so that :} \qquad [-1,1]^2 \ = \ \bigcup_{(i_1,i_2)\in\Sigma^2} Q_{(i_1,i_2)}(\delta)$$

that we order following the lexicographic order of Σ^2.

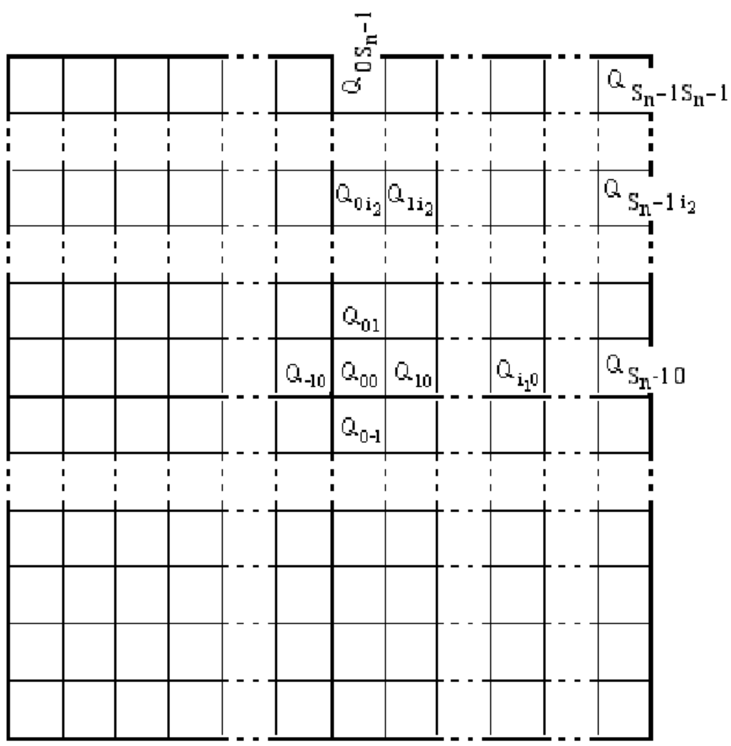

Figure 7

Let us denote by $T_n^r(x_0)$ the r-th barycentric subdivision of the triangulation $T_n(x_0)$ of $F_n = \pi_{XY}^{-1}(x_0) \cap A_n$ and consider for each closed simplex $\sigma \in T_n^r(x_0)$ its open simplicial neighbourhood $N(\sigma)$ in $T_n^r(x_0)$.

Because

$$\lim_{\delta\to 0} diam \, Q_{(i_1,i_2)}(\delta) = 0 \quad \text{and} \quad \lim_{r\to\infty} diam N(\sigma) = 0$$

and H is a Lipschitz map on the compact set A_n, we have :

$$\lim_{(\delta,r)\to(0,+\infty)} diam \; H_{x_0}\big(Q_{(i_1,i_2)}(\delta) \, \times \, N(\sigma)\big) \quad = \quad 0 \, .$$

There exists then $\delta > 0$ small enough and $r \in \mathbb{N}$ big enough such that for each $\sigma \in T_n^r(x_0)$ $(\dim \sigma = m)$, the diameter of $H\left(Q_{(i_1,i_2)}(\delta) \, \times \, N(\sigma)\right)$ is smaller than $\delta < \eta_n^0$, so that it is contained in an open set $V_{z_j} = H_{x_j}(Q_0(\delta_{z_j}) \times J_{z_j})$ of the covering $\mathcal{C}_n$.

Moreover, we can ensure that each $diam\, Q_{(i_1,i_2)}(\delta) < \eta_n^1$ so that each $Q_{(i_1,i_2)}(\delta) \subseteq I_{x_j}$ and each $diam\, N(\sigma) < \eta_n^2$ so that $N(\sigma) \subseteq J_{z_j}^0$ for some j.

Hence the foliations $\mathcal{H}_{x_j} = \left\{ M_z^{x_j} := H_{x_j}\left(Q_0(\delta_{z_j}) \times \{z\}\right)\right\}_{z \in J_{z_j}}$, which are $\frac{1}{n}$-close to the canonical distribution $\mathcal{D}_{XY}$, fill the open sets $V_{z_j} = H_{x_j}(Q_0(\delta_{z_j}) \times J_{z_j})$ containing the sets $H\left(Q_{(i_1,i_2)}(\delta) \times N(\sigma)\right)$.

On the other hand, A_n is obviously covered by the images :

$$\begin{aligned} A_n &= H\Big(\bigcup_{(i_1,i_2)\in\Sigma^2} Q_{(i_1,i_2)}(\delta) \quad \times \bigcup_{\sigma \in T_n^r(x_0)} N(\sigma)\Big) \\ &= \bigcup_{(i_1,i_2)\in\Sigma^2} \quad \bigcup_{\sigma\in T_n^r(x_0)} H\Big(Q_{(i_1,i_2)}(\delta) \;\times\; N(\sigma)\Big). \end{aligned}$$

We will show how the foliations $\mathcal{H}_{x_j}$ can be glued together to obtain a convenient foliation of the whole of A_n which is again $\frac{1}{n}$-close to the distribution $\mathcal{D}_{XY}$.

This proof will take several steps.

Step 1 : For every $(i_1, i_2) \in \Sigma^2$, there exists a (π, ρ)-controlled frame field generating a foliation $\mathcal{H}_{(i_1,i_2)}$, $\frac{1}{n}$-close to $\mathcal{D}_{XY}$, of an open neighbourhood of :

$$\pi_{XY}^{-1}\big(Q_{(i_1,i_2)}\big) \cap A_n \;=\; H\left(Q_{(i_1,i_2)} \times \bigcup_{\sigma\in T_n^r(x_0)} N(\sigma)\right), \qquad \text{where } Q_{(i_1,i_2)} := Q_{(i_1,i_2)}(\delta)\,.$$

Fix $(i_1, i_2) \in \Sigma^2$ and recall that $P_n := \{z_1, \ldots, z_{q_n}\}$ is the finite set of points z_j whose neighbourhoods V_{z_j} cover A_n and are foliated by the foliations $\mathcal{H}_{x_j}$ (with $x_j = \pi_X(z_j)$) which are $\frac{1}{n}$-close to $\mathcal{D}_X$. Let us write

$$P_n(i_1,i_2) = \left\{ z_j \in P_n \mid \exists\, \sigma \in T_n^r(x_0) \;:\; H\left(Q_{(i_1,i_2)} \times N(\sigma)\right) \subseteq V_{z_j} \right\}$$

and remark that :

$$\begin{aligned} \pi_{XY}^{-1}\big(Q_{(i_1,i_2)}\big) \cap A_n \;&=\; H\Big(\; Q_{(i_1,i_2)} \,\times\, F_n\Big) \\ &\subseteq \bigcup_{\sigma\in T_n^r(x_0)} H\Big(\; Q_{(i_1,i_2)} \,\times\, N(\sigma)\Big) \;\subseteq\; \bigcup_{z_j\in P_n(i_1,\, i_2)} V_{z_j}\,. \end{aligned}$$

Now $p = (i_1\delta,\, i_2\delta) \in Q_{(i_1,i_2)}$ is the first vertex (with the lexicographic order) and we have a restriction homeomorphism

$$H_{|} \;:\; \left(\pi_{XY}^{-1}(x_0),\, x_0\right) \to \left(\pi_{XY}^{-1}(p)\,, p\right) \quad , \quad H(y_0) = H(i_1\delta,\, i_2\delta,\, y_0)$$

and hence the triangulation $T_n^r(x_0)$ of $F_n = \pi_{XY}^{-1}(x_0) \cap A_n$ induces naturally a triangulation

$$T_n^r(p) := H(\{p\} \times T_n^r(x_0)) \quad \text{of} \quad F_n(p) = \pi_{XY}^{-1}(p) \cap A_n$$

and an open cellular covering of $F_n(p)$ given by

$$\mathcal{R}_n(p) \;=\; \Big\{ N_p(\sigma) := H\big(\{p\} \times N(\sigma)\big) \quad \Big| \quad \sigma \in T_n^r(x_0)\,,\; \dim\sigma = k-2 \Big\}\,.$$

Let $\mathcal{P} = \big\{\alpha_\sigma : N_p(\sigma) \to [0,1] \;\big|\; \sigma \in T^r_n(p)\big\}$ be a smooth partition of unity subordinate to the covering $\mathcal{R}_n(p)$.

Because each leaf $M^{x_j}_z$ of the foliation $\mathcal{H}_{x_j} = \{M^{x_j}_z\}_{z\in J_{z_j}}$ meets the fiber $F_n(p) = \pi^{-1}_{XY}(p)$ in a unique point

$$\{\, z^p_j \,\} \;:=\; M^{x_j}_z \cap \pi^{-1}_{XY}(p) \;=\; H_{x_j}(p - x_j, z) \qquad \text{with} \quad z \in J_{z_j}\,,$$

for every $y = H_{x_j}(t_1, t_2, z) \in V_{z_j} = H_{x_j}(Q_0(\delta_{z_j}) \times J_{z_j})$, the point z^p_j is the "horizontal projection" of z and y on $\pi^{-1}_{XY}(p)$ via the restriction $H_{x_j|} : \pi^{-1}_{XY}(x_j) \to \pi^{-1}_{XY}(p)$, and lies in a unique open simplex $\sigma^o \in T^r_n(p)$ and allows us to define an "adapted" partition of unity, $\tilde{\mathcal{P}}$ subordinate to the covering $\{V_{z_j}\}_{z_j\in P_n(i_1,i_2)}$ by extending it *constant along each leaf of* $\mathcal{H}_{x_j}$. That is we define : $\tilde{\mathcal{P}} = \{\tilde\alpha_j : V_{z_j} \to [0,1]\}_{z_j\in P_n(i_1,i_2)}$ as follows :

$$\tilde\alpha_j \;:\; V_{z_j} \longrightarrow [0,1] \qquad \text{by} \qquad \tilde\alpha_j(y) = \alpha_\sigma(z^p_j) \qquad \text{where } y \in \sigma^o \in T^r_n(p)\,.$$

Now we use this partition of unity to glue together all the (π,ρ)-controlled frame fields $\{(w^{x_j}_1, w^{x_j}_2)\}_j$ of the foliations $\mathcal{H}_{x_j}$ to define on the open set

$$\bigcup_{z_j\in P_n(i_1,i_2)} V_{z_j} \;\supseteq\; \bigcup_{\sigma\in T^r_n(x_0)} H\Big(\, Q_{(i_1,i_2)} \times N(\sigma)\Big) \;\supseteq\; \pi^{-1}_{XY}\big(Q_{(i_1,i_2)}\big) \cap A_n$$

the new frame field :

$$W^{(i_1,i_2)}_1(y) \;=\; \sum_{z_j\in P_n(i_1,i_2)} \tilde\alpha_j(y)\cdot w^{x_j}_1(y) \qquad \text{and} \qquad W^{(i_1,i_2)}_2 \;=\; v_2\,.$$

Then the Lie bracket :

$$\big[\, W^{(i_1,i_2)}_1 ,\; W^{(i_1,i_2)}_2 \,\big](y) = \Big[\sum_{j\in P_n(i_1,i_2)} \tilde\alpha_j w^{x_j}_1 ,\; v_2\Big](y) =$$

$$\sum_{j\in P_n(i_1,i_2)} \Big(\tilde\alpha_{j*y}(v_2(y))\cdot w^{x_j}_1(y) \;+\; \tilde\alpha_j(y)\cdot[w^{x_j}_1, w^{x_j}_2]\Big) \;=\; \sum_{j\in P_n(i_1,i_2)} (0+0) = 0$$

where each $\tilde\alpha_{j*y}(v_2(y)) = 0$ because the $\tilde\alpha_j$ are constant along the trajectories of v_2 and each $[w^{x_j}_1, v_2](y) = [w^{x_j}_1, w^{x_j}_2](y) = 0$ because $(w^{x_j}_1, w^{x_j}_2)$ is a generating frame field of the foliation $\mathcal{H}_{x_j}$.

On the other hand, each $(w^{x_j}_1, w^{x_j}_2)$ being π_{XY}-controlled, we have :

$$\pi_{X*y}\big(W^{(i_1,i_2)}_1(y)\big) \;=\; \pi_{X*y}\big(\sum_j \tilde\alpha_j(y) w^{x_j}_1(y)\big) \;=\; \sum_j \tilde\alpha_j(y)\cdot\pi_{X*y}\big(w^{x_j}_1(y)\big) \;=\; 1\cdot(E_1) \;=\; E_1$$

and similarly each $(w^{x_j}_1, w^{x_j}_2)$ being ρ_X-controlled, we have :

$$\rho_{X*y}\big(W^{(i_1,i_2)}_1(y)\big) \;=\; \rho_{X*y}\big(\sum_j \tilde\alpha_j(y) w^{x_j}_1(y)\big) \;=\; \sum_j \tilde\alpha_j(y)\cdot\rho_{X*y}\big(w^{x_j}_1(y)\big) \;=\; \sum_j \tilde\alpha_j(y)\cdot 0 \;=0\,.$$

Thus the frame field $(W^{(i_1,i_2)}_1, W^{(i_1,i_2)}_2) = (W^{(i_1,i_2)}_1, v_2)$ is (π,ρ)-controlled too.

This means, in particular, that the partition of unity modifies *only* the components of $W_1^{(i_1,i_2)}(y)$ along the tangent space to the link $L(y)$ of the π_{XY}-fibre containing y.

Finally, each $\mathcal{H}_{x_j}$ being $\frac{1}{n}$-close to $\mathcal{D}_{XY}$ one finds :

$$||W_1^{(i_1,i_2)}(y) - v_1(y)|| = \Big|\Big| \sum_j \tilde{\alpha}_j(y)\big(w_1^{x_j}(y) - v_1(y)\big) \Big|\Big| \leq$$

$$\leq \sum_j \tilde{\alpha}_j(y) \cdot ||w_1^{x_j}(y) - v_1(y)|| \leq \sum_j \tilde{\alpha}_j(y) \cdot \left(\frac{1}{n}\right) = 1 \cdot \left(\frac{1}{n}\right) = \frac{1}{n} .$$

Hence the (π, ρ)-controlled frame field $(W_1^{(i_1,i_2)}, W_2^{(i_1,i_2)}) = (W_1^{(i_1,i_2)}, v_2)$ generates a new foliation $\mathcal{H}_{(i_1,i_2)}$ on an open set containing $\pi_{XY}^{-1}\big(Q_{(i_1,i_2)}\big) \cap A_n$ on which it is $\frac{1}{n}$-close to the canonical distribution $\mathcal{D}_{XY}$.

We will denote by $H_{(i_1,i_2)} : \mathbb{R}^2 \times \pi_{XY}^{-1}(p) \to Y$ the induced topological trivialization obtained by composing the flows of such frame fields. Then $H_{(i_1,i_2)}$ also generates $\mathcal{H}_{(i_1,i_2)}$.

Step 2 : For each fixed i_2, there exists a foliation $\mathcal{H}_{i_2}$, $\frac{1}{n}$-close to $\mathcal{D}_{XY}$ on an open neighbourhood of :

$$\bigcup_{i_1 \in \Sigma} \pi_{XY}^{-1}\big(Q_{(i_1,i_2)}\big) \cap A_n = \pi_{XY}^{-1}\big([-1,1] \times Q_{i_2}\big) \cap A_n .$$

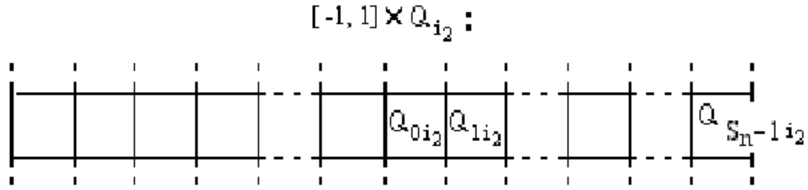

Figure 8

Fix an $i_2 \in \Sigma$ and, for every $i_1 \in \Sigma$, consider the foliation $\mathcal{H}_{(i_1,i_2)}$ obtained in step 1 with generating frame field $(W_1^{(i_1,i_2)}, W_2^{(i_1,i_2)}) = (W_1^{(i_1,i_2)}, v_2)$.

We will show how the foliations $\mathcal{H}_{(0,i_2)}$ and $\mathcal{H}_{(1,i_2)}$ glue together to give a new foliation $\mathcal{H}_{((0,1),i_2)} := \mathcal{H}_{(0,i_2)} \vee \mathcal{H}_{(1,i_2)}$ of an open neighbourhood of

$$\Big(\pi_{XY}^{-1}\big(Q_{(0,i_2)}\big) \cup \pi_{XY}^{-1}\big(Q_{(1,i_2)}\big)\Big) \cap A_n = \pi_{XY}^{-1}\big([0,2\delta] \times Q_{i_2}\big) \cap A_n .$$

Let α be a smooth decreasing function :

$$\alpha : [0,2\delta] \to [0,1] \quad \text{such that} \quad \alpha(t) = \begin{cases} 1 & \text{if} \quad t \in [0, \frac{1}{2}\delta] \\ 0 & \text{if} \quad t \in [\frac{3}{2}\delta, 2\delta] . \end{cases}$$

Then α can be extended to a map defined on a neighbourhood of $\pi_{XY}^{-1}\big([0,2\delta] \times Q_{i_2}\big) \cap A_n$ which is constant on the trajectories of v_2. That is for every $y = H_{(0,i_2)}(t_1, t_2, y_0)$ we define:

$$\tilde{\alpha} : \pi_{XY}^{-1}\big([0,2\delta] \times Q_{i_2}\big) \cap A_n \longrightarrow [0,1] \quad , \quad \tilde{\alpha}(y) = \tilde{\alpha}\big(H_{(0,i_2)}(t_1,t_2,y_0)\big) = \alpha(t_1) .$$

We consider the vector field

$$W_1^{((0,1),i_2)}(y) \;=\; \tilde{\alpha}(y)\cdot W_1^{(0,i_2)}(y) \;+\; \big(1-\tilde{\alpha}(y)\big)\cdot W_1^{(1,i_2)}(y)$$

where the verifications that the Lie bracket $[W_1^{((0,1)\,,\,i_2)},\; v_2\,](y) \;=\; 0$ and that $W_1^{((0,1),i_2)}$ is a (π,ρ)-controlled vector field $\frac{1}{n}$-close to $\mathcal{D}_X$ are similar to and simpler than those seen in step 1.

Continuing in this way, after a finite number of steps we obtain a vector field $W_1^{i_2}(y)$ defined on a neighbourhood of $\pi_{XY}^{-1}\big([-1,1]\times Q_{i_2}\big)\cap A_n$.

Remark 6. In the construction of the final vector field $W_1^{i_2}(y)$ of step 2, for example when we glue $W_1^{((0,1),i_2)}(y)$ to $W_1^{(2,i_2)}(y)$, the new partition of unity will act only for values of $t_1\in[\frac{3}{2}\delta,\frac{5}{2}\delta]$ so as to give a vector field $W_1^{((0,1,2),i_2)}(y)$ defined on $\pi_{XY}^{-1}\big([0,3\delta]\times Q_{i_2}\big)\cap A_n$ and satisfying :

$$W_1^{((0,1,2),i_2)}(y) \;=\; \begin{cases} W_1^{((0,1),i_2)}(y) & \text{for} \quad t_1\in[0,\frac{3}{2}\delta] \\[2ex] W_1^{(2,i_2)}(y) & \text{for} \quad t_1\in[\frac{5}{2}\delta,3\delta]\,. \end{cases}$$

Hence this second gluing is in a set <u>disjoint</u> from the set in which we did the first gluing and this ensures that $||W_1^{((0,1,2),i_2)}(y)-v_1(y)||\le\frac{1}{n}$.

This argument holding for all successive gluing one obtains a final vector field :

$$W_1^{i_2}(y) \;:=\; W_1^{((-s_n,\ldots,s_{n-1}),i_2)}(y) \qquad \text{satisfying} \qquad ||W_1^{i_2}(y)-v_1(y)||\le\frac{1}{n}\,. \quad \square$$

Therefore, the final commuting frame field $(W_1^{i_2},v_2)$ generates the foliation claimed in step 2 :

$$\mathcal{H}_{i_2} \;:=\; \mathcal{H}_{(-s_n,i_2)}\vee\;\ldots\vee\;\mathcal{H}_{(0,i_2)}\vee\;\ldots\;\vee\mathcal{H}_{(s_n-1,i_2)}$$

which is $\frac{1}{n}$-close to $\mathcal{D}_X$ on an open neighbourhood of $\pi_{XY}^{-1}\big([-1,1]\times Q_{i_2}\big)\cap A_n$.

We will denote by H_{i_2} the induced topological trivialization obtained by composing the flows of this frame field $(W_1^{i_2},v_2)$ and generating $\mathcal{H}_{i_2}$.

Step 3 : There exists a foliation $\mathcal{F}_n$ and its (π,ρ)-controlled frame field (W_1,v_2) which is $\frac{1}{n}$-close to $\mathcal{D}_{XY}$ on an open neighbourhood of :

$$\pi_{XY}^{-1}\big([-1,1]^2\big)\cap A_n \;=\; \bigcup_{i_2\in\Sigma}\pi_{XY}^{-1}\big([-1,1]\times Q_{i_2}\big)\cap A_n\,.$$

Let us fix $i_2\in\{0,1\}$. We will show below how the foliations $\mathcal{H}_0$ and $\mathcal{H}_1$ and their generating frame fields (W_1^0,v_2) and (W_1^1,v_2) glue together to obtain a convenient foliation $\mathcal{H}_0\vee\mathcal{H}_1$ of an open neighbourhood of

$$\pi_{XY}^{-1}\big([-1,1]\times Q_0\big)\cup\pi_{XY}^{-1}\big([-1,1]\times Q_1\big)\cap A_n \;=\; \pi_{XY}^{-1}\big([-1,1]\times[0,2\delta]\big)\cap A_n\;.$$

Let α be the smooth decreasing function of step 2. This time we cannot extend α to be constant along the t_2-trajectories (because the gluing must be done along the t_2-trajectories), so we extend it to be constant on the t_1-trajectories.

Hence we define a map on a neighbourhood of $\pi_{XY}^{-1}\big([-1,1]\times[0,2\delta]\big)\cap A_n$ by setting for every $y = H_1(t_1,t_2,y_0)$:

$$\beta \ : \ \pi_{XY}^{-1}\big([-1,1]\times[0,2\delta]\big)\cap A_n \longrightarrow [0,1] \quad , \quad \beta(y) = \beta(H_1(t_1,t_2,y_0)) = \alpha(t_2)\,.$$

We define the vector field W_1 by :

$$W_1(y) \ := \ \beta(y)\cdot W_1^0(y) + \big(1-\beta(y)\big)\cdot W_1^1(y)$$

for which the verification that it is (π,ρ)-controlled is similar and simpler in than step 1.

It is easy to see that W_1 is again $\frac{1}{n}$-close to v_1 :

$$\begin{aligned} ||W_1(y)-v_1(y)|| \ &\leq \ \beta(y)\cdot||W_1^0(y)-v_1(y)|| \ + \ \big(1-\beta(y)\big)\cdot||W_1^1(y)-v_1(y)\ || \\ &\leq \ \beta(y)\cdot\frac{1}{n} \ + \ \big(1-\beta(y)\big)\cdot\frac{1}{n} \ = \ \frac{1}{n}\,. \end{aligned}$$

However unfortunately this time the Lie bracket

$$\big[W_1\,,\,v_2\big](y) \ = \ \big[\ \beta\cdot W_1^0 + \big(1-\beta\big)\cdot W_1^1\ ,\ v_2\big](y)$$

$$= \ \Big(\beta_{*y}(v_2(y))\cdot W_1^0(y) \ + \ \beta(y)\cdot[W_1^1,\,v_2](y)\Big) \ - \Big(\beta_{*y}(v_2(y))\cdot W_1^0(y) \ + \ \beta(y)\cdot[W_1^1,\,v_2](y)\Big)$$

$$= \ \beta_{*y}(v_2(y))\cdot W_1^0(y) \ + \ 0 \ - \beta_{*y}(v_2(y))\cdot W_1^1(y) \ + \ 0$$

$$= \ \beta_{*y}(v_2(y))\cdot\big(W_1^0(y)-W_1^1(y)\big)$$

is not zero in general.

But W_1 being (π,ρ)-controlled, and $\frac{1}{n}$-close to lifting v_1 on $\mathcal{D}_{XY}$ we can use the flow ψ_1 of W_1 and the flows ϕ_2 of v_2 to define the desired new foliation $\mathcal{H}_0\vee\mathcal{H}_1$ on a neighbourhood of $\pi_{XY}^{-1}\big([-1,1]\times[0,2\delta]\big)\cap A_n$ as follows.

We define :

$$\begin{aligned} K \ : \quad \Big([-1,1]\times[0,2\delta]\Big)\cap F_n(x_0) \quad &\longrightarrow \quad \pi_{XY}^{-1}\big([-1,1]\times[0,2\delta]\big) \\ (t_1,t_2,y_0) \quad &\longrightarrow \quad \phi_2(t_2,\ \psi_1(t_1,y_0))\ . \end{aligned}$$

where ϕ_2 is the flow of v_2 and it is easy to verify that the foliation

$$\mathcal{H}_0\vee\mathcal{H}_1 := \Big\{K\big([-1,1]\times[0,2\delta]\times\{y_0\}\big)\ \Big\}_{y_0\in F_n(x_0)}$$

has a generating frame field $(\widetilde{W}_1,v_2)$ where

$$\widetilde{W}_1(y) =: \widetilde{W}_1^{(0,1)}(y) := K_{*(t_1,t_2,y_0)}(E_1) \ = \ \phi_{2*y_1}^{t_2}(W_1(y_1))\,.$$

Recalling that $W_1(y) := \beta(y)\cdot W_1^0(y) + \big(1-\beta(y)\big)\cdot W_1^1(y)$, $\widetilde{W}_1$ satisfies :

$$\begin{aligned} (*) : \ ||\widetilde{W}_1(y)-v_1(y)|| \ &= \ \big|\big|\phi_{2*y_1}^{t_2}(W_1(y_1))-v_1(y)\big|\big| \\ &= \ \Big|\Big|\phi_{2*y_1}^{t_2}\Big(\beta(y)\cdot W_1^0(y_1) \ + \ \big(1-\beta(y)\big)\cdot W_1^1(y_1)\Big)-v_1(y)\Big|\Big| \end{aligned}$$

$$\leq \beta(y)\cdot\big|\big|\phi^{t_2}_{2*y_1}\big(W^0_1(y_1)\big)-v_1(y)\big|\big| + \big(1-\beta(y)\big)\cdot\big|\big|\phi^{t_2}_{2*y_1}\big(W^1_1(y_1)\big)-v_1(y)\big|\big|$$
$$\leq \beta(y)\cdot\big|\big|W^0_1(y)-v_1(y)\big|\big| + \big(1-\beta(y)\big)\cdot\big|\big|W^1_1(y)-v_1(y)\big|\big|$$

and by the inequalities obtained at the end of Remark 6 in step 2 for $W_1^{i_2}(y)$ for every $i_2 = -s_n, \ldots, s_{n-1}$ applied to $W_1^0(y)$ and $W_1^1(y)$ we find

$$\leq \beta(y)\cdot\frac{1}{n} + \big(1-\beta(y)\big)\cdot\frac{1}{n} = \frac{1}{n}\,.$$

This proves that the vector field $\widetilde{W}_1(y) = \widetilde{W}_1^{(0,1)}$ is again $\frac{1}{n}$-close to $\mathcal{D}_X$.

At the second gluing, we define a vector field :

$$\widetilde{W}_1^{(0,1,2)}(y) := K^{(0,1,2)}_{*(t_1,t_2,y_0)}(E_1) = \phi^{t_2}_{2*y_1}(W_1^2(y_1))\,.$$

which satifies :

$$||\widetilde{W}_1^{(0,1,2)}(y) - v_1(y)|| \leq \frac{1}{n} \qquad \text{exactly as in (*) of Step 3,}$$

with a formal repetition of the inequalities $(*)$ in which we replace $\widetilde{W}_1^{(0,1)}(y) = \widetilde{W}_1(y)$ by $\widetilde{W}_1^{(0,1,2)}(y)$ etc . . . and at the end, this time, using that :

$$||\widetilde{W}_1^{(0,1)}(y) - v_1(y)|| \leq \frac{1}{n} \quad \text{and} \quad ||W_1^2(y) - v_1(y)|| \leq \frac{1}{n}\,.$$

Continuing in this way, after $2s_n - 1$ steps we define a vector field

$$u_1^n(y) := \widetilde{W}_1^{(-s_n,\ldots,s_{n-1})} \qquad \text{on a neighbourhood of} \quad \pi^{-1}_{XY}\big([-1,1]\times[-1,1]\big)\cap A_n$$

such that the frame field (u_1^n, v_2) is (π,ρ)-controlled and generates the desired foliation

$$\mathcal{F}_n : \quad \mathcal{H}_{-s_n} \vee \ldots \vee \mathcal{H}_0 \vee \mathcal{H}_1 \vee \ldots \vee \mathcal{H}_{s_n-1}$$

which, with the same arguments as in Remark 6, where this time we glue the foliations $\mathcal{H}_j$ along the i_2-direction instead of the i_1-direction, one checks to be $\frac{1}{n}$-close to the canonical distribution $\mathcal{D}_{XY}$.

Step 4 : There exists a global foliation on a neighbourhood $\pi^{-1}_{XY}\big([-1,1]^2\big)$ of x_0 in $X \sqcup Y$ and this proves Theorem 4 for $l = 2$.

In the previous step we constructed for every $n \in \mathbb{N}^*$, a foliation $\mathcal{F}_n$ and its generating (π,ρ)-controlled frame field (u_1^n, v_2) which is $\frac{1}{n}$-close to $\mathcal{D}_X$ on an open neighbourhood of the solid annulus $\pi^{-1}_{XY}\big([-1,1]\times[-1,1]\big)\cap A_n$.

We prove now that all foliations of the sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}^*}$ glue together to give a final foliation $\mathcal{H}$ defined on the whole of $\pi^{-1}_{XY}\big([-1,1]^2\big) \equiv W = Y$ satisfying the smooth version of the Whitney fibering conjecture.

Fix $n \geq 1$ and consider the two foliations :

$$\begin{cases} \mathcal{F}_n & \text{with generating frame } (u_1^n, v_2) \quad \frac{1}{n}\text{-close to } \mathcal{D}_X \quad \text{on } A_n \subseteq \rho^{-1}_{XY}\big([\frac{1}{n+2}, \frac{1}{n}]\big) \\ \text{and} & \\ \mathcal{F}_{n+1} & \text{with generating frame } (u_1^{n+1}, v_2) \quad \frac{1}{n+1}\text{-close to } \mathcal{D}_X \quad \text{on } A_{n+1} \subseteq \rho^{-1}_{XY}\big([\frac{1}{n+3}, \frac{1}{n+1}]\big). \end{cases}$$

Let α_1 be a smooth increasing function,

$$\alpha : [\tfrac{1}{n+3}, \tfrac{1}{n}] \to [0,1] \quad \text{such that} \quad \alpha_1(t) \ = \ \begin{cases} 0 & \text{if} \quad t \in [\frac{1}{n+3} \, , \frac{1}{n+2}] \\ \\ 1 & \text{if} \quad t \in [\frac{1}{n+1} \ , \ \frac{1}{n}] \, . \end{cases}$$

By using the function α_1 we will glue together the foliations $\mathcal{F}_n$ and $\mathcal{F}_{n+1}$ along their intersection $A_n \cap A_{n+1} \subseteq \rho^{-1}_{XY}\big([\frac{1}{n+2}, \frac{1}{n+1}]\big)$ without changing them in $\rho^{-1}_{XY}\big([\frac{1}{n+3}, \frac{1}{n+2}] \cup [\frac{1}{n+1}, \frac{1}{n}]\big)$.

Consider the vector field $w_1^{n+1} \ : \ A_n \cup A_{n+1} \to \mathbb{R}^k$ defined by :

$$w_1^{n+1}(y) \ = \ \gamma(y) \cdot u_1^n(y) \ + \ (1-\gamma(y)) \cdot u_1^{n+1}(y) \quad \text{where} \quad \gamma(y) = \alpha_1 \circ \rho_{XY}(y)$$

which coincides with $u_1^n(y)$ for $y \in \rho^{-1}_{XY}([\frac{1}{n+1}, \frac{1}{n}])$.

We have :

$$[w_1^{n+1}, v_2](y) \ = \ \Big(\gamma_{*y}(v_2(y)) \cdot u_1^n(y) \ + \ \gamma(y) \cdot [u_1^n, v_2](y)\Big)$$

$$- \Big(\gamma_{*y}(v_2(y)) \cdot u_1^{n+1}(y) \ + \ \gamma(y) \cdot [u_1^{n+1}, v_2](y)\Big) = 0 - 0 = 0$$

where $\gamma_{*y}(v_2(y)) = 0$ because $\gamma(t)$ is constant along all the trajectories of v_2 and each $[u_1^n, v_2](y) = [u_1^{n+1}, v_2](y) = 0$ because (u_1^n, v_2) and (u_1^{n+1}, v_2) are two generating frame fields respectively of the foliations $\mathcal{F}_n$ and $\mathcal{F}_{n+1}$.

Hence, the frame field (w_1^{n+1}, v_2) defines a new foliation $\mathcal{F}_n \vee \mathcal{F}_{n+1}$ on $A_n \cup A_{n+1}$ for which it is easy to verify that (w_1^{n+1}, v_2) is (π, ρ)-controlled and coincides with $\mathcal{F}_n$ on the upper part $A_n \cap \rho^{-1}_{XY}([\frac{1}{n+1}, \frac{1}{n}])$ of A_n .

Moreover $\mathcal{F}_n \vee \mathcal{F}_{n+1}$ is $\frac{1}{n}$-close to $\mathcal{D}_{XY}$ in $A_n \cup A_{n+1}$:

$$||w_1^{n+1}(y) - v_1(y)|| \ \leq \quad \gamma(y) \cdot ||u_1^n(y) - v_1(y)|| \ + \ (1-\gamma(y)) \cdot ||u_1^{n+1}(y) - v_1(y) \, ||$$

$$\leq \ \gamma(y) \cdot \frac{1}{n} \ + \ (1-\gamma(y)) \cdot \frac{1}{n+1} \ \leq \ \frac{1}{n} \, .$$

Using this way of gluing together inductively the foliations of the sequence $\{\mathcal{F}_n\}_{n \geq 1}$ starting from $\mathcal{H}_1 := \mathcal{F}_1 \vee \mathcal{F}_2$ we define an "increasing" sequence of foliations $\{\mathcal{H}_n\}_{n \geq 1}$:

$$\mathcal{H}_n := \Big(\big((\mathcal{F}_1 \vee \mathcal{F}_2\big) \vee \ldots \vee \mathcal{F}_n\Big) \vee \mathcal{F}_{n+1} \ \text{ of the annular region } \quad A_1 \cup \ldots \cup A_{n+1} \ \subseteq \ \rho^{-1}_{XY}([\tfrac{1}{n+3} \ , \ 1])$$

where $\mathcal{H}_n$ coincides with $\mathcal{H}_{n-1}$ on $\rho^{-1}_{XY}([\frac{1}{n} \ , \ 1])$ and is $\frac{1}{n-1}$-close to $\mathcal{D}_X$ on A_n.

In this way the restrictions $\mathcal{H}'_n \ := \ \mathcal{H}_{n|\rho^{-1}_{XY}([\frac{1}{n} \ , \ 1])}$ define an increasing sequence of foliations with each $\mathcal{H}'_n$ which is $\frac{1}{n-1}$-close to $\mathcal{D}_X$ on A_n.

By considering on the whole of $W = \pi^{-1}_{XY}([-1,1]^2 \cap \rho^{-1}_{XY}(]0,1])$ the foliation union $\mathcal{H}' = \cup_{n=1}^{\infty} \mathcal{H}'_n$ and using that, by (c)-regularity $[\mathbf{MT}]_2$, $\lim_{y \to x \in X} \mathcal{D}_{XY}(y) = T_x X$ for every $x \in X$, we conclude that :

$$\lim_{y \to x \in X} T_y \mathcal{H} \ = \ \lim_{y \to x \in X} \mathcal{D}_{XY}(y) \ = \ T_x X \, .$$

Step 5 : The general case with $\dim X = l \geq 2$.

The proof of Theorem 4 in the general case with $\dim X = l \geq 2$ can be obtained directly by a formal repetition of the steps 1 to 4 of the proof of the case $\dim X = 2$ where the paving by squares $\left\{ Q_{(i_1,i_2)}(\delta) := Q_{i_1} \times Q_{i_2} \right\}_{(i_1,i_2)\in\Sigma^2}$ of $[-1,1]^2$ is replaced by a paving by l-cubes $\{Q_{\underline{i}}(\delta) := Q_{i_1} \times \ldots \times Q_{i_l}\}_{\underline{i}\in\Sigma^l}$ of $[-1,1]^l$ using a multi-index $\underline{i} = (i_1, \ldots, i_l)$ and all essential ideas and techniques are adapted to a bigger dimension.

However this would be long and formally heavy so we give a shorter inductive proof.

Theorem 4 was proved for $\dim X = l = 2$.

Let $\dim X = l > 2$ and suppose Theorem 3 is true for all X' such that $\dim X' = l - 1$.

Let $Y > X$. In a local analysis we suppose as usual $X = \mathbb{R}^l \times \{0^{n-l}\}$, $x_0 = 0^n \in X$.

For every $t \in [-1,1]$, let $X_t = \mathbb{R}^{l-1} \times \{t\} \times \{0^{n-l}\}$ and $Y_t := \pi_{XY}^{-1}(X_t)$, then $X_t < Y_t$ is a (c)-regular stratification with control data $(\pi_{X_t}, \rho_{X_t}) : X_t \cup Y_t \to X_t \times [0,1]$ which is the restriction of the control data $(\pi_X, \rho_X) : X \cup Y \to X \times [0,1]$ of $X \cup Y$.

The stratification $X_t \cup Y_t$ satisfying the inductive hypothesis we assume for it all results obtained in the previous steps $1, \ldots, 4$, starting from a topological trivialisation H_{0_t} of origin $0_t := (0^{l-1}, t)$ and a canonical distribution $\mathcal{D}_{X_tY_t}(y) = [v_1(y), \ldots, v_{l-1}(y)]$ generated by the first $l-1$ coordinates of the frame field, which is a continuous controlled canonical lifting $(v_1(y), \ldots, v_l(y))$ of the standard frame field $(E_1, \ldots, E_l)$, and which generates the canonical distribution $\mathcal{D}_{XY}(y) = [v_1(y), \ldots, v_l(y)]$ of X.

By inductive hypothesis every pair of strata $X_t < Y_t$ admits an (a)-regular $(l-1)$-foliation $\mathcal{H}_t = \{M_{y_{0_t}} := H_t(\mathbb{R}^{l-1} \times \{y_{0_t}\})\}_{y_{0_t}\in\pi_X^{-1}(0_t)}$ of Y_t obtained from a trivialisation

$$H_t : \mathbb{R}^{l-1} \times \pi_{X_t}^{-1}\{0_t\} \to Y_t \qquad \text{where} \qquad y_{0_t} \in \pi_{X_t}^{-1}(0_t)\,.$$

Moreover following our proof in step 4, by induction every foliation $\mathcal{H}_t$ is $\frac{1}{n}$-close to $\mathcal{D}_{X_tY_t}$ in the annulus $A_{n+1,t} := A_{n+1} \cap Y_t$.

Let $y_{l-1,t}$ denote an arbitrary point of Y_t.

For every $t \in [-1,1]$ the frame field $(u_1^t(y), \ldots, u_{l-1}^t(y))$ defined by

$$u_i^t(y_{l-1,t}) \;:=\; H_{t*(t_1,\ldots,t_{l-1},y_{0_t})}(E_i) \qquad \text{for every} \qquad i = 1, \ldots, l-1$$

is, by Proposition 3, §5.2, the unique commuting (π_{X_t}, ρ_{X_t})-controlled frame field tangent to $\mathcal{H}_t$, generating $T_{y_{l-1,t}}\mathcal{H}_t$ and is $\frac{1}{n}$-close to $\mathcal{D}_{X_tY_t}$ (and $\mathcal{D}_{XY}$) in the annulus $A_{n+1,t}$ and continuous on X_t (step 4).

Moreover one can write ([**Mu**]$_1$, Chap 2, §5.2 Prop. 1) :

$$\begin{array}{rcccl} H_t & : & \mathbb{R}^{l-1} \times \pi_{X_t}^{-1}\{0_t\} & \longrightarrow & \pi_{X_t}^{-1}(X_t) \equiv Y_t \\[2mm] & & (t_1, \ldots, t_{l-1}, y_{0_t}) & \longmapsto & y_{l-1,t} := \psi_{l-1}^t(t_{l-1}, \ldots, \psi_1^t(t_1, y_{0_t}) \ldots) \end{array}$$

where $(\psi_1^t, \ldots, \psi_{l-1}^t)$ are the commuting flows of the frame field $(u_1^t, \ldots, u_{l-1}^t)$.

Each map H_t extends in a natural way along the direction of the vector field v_l using its flow ϕ_l by setting

$$\begin{array}{rcccl} H^t & : & \mathbb{R}^{l} \times \pi_{X_t}^{-1}\{0_t\} & \longrightarrow & \pi_{X}^{-1}(X) \equiv Y \\[2mm] & & (t_1, \ldots, t_{l-1}, t_l, y_{0_t}) & \longmapsto & y_{l,t} \;:=\; \phi_l(t_l, \psi_{l-1}^t(t_{l-1}, \ldots, \psi_1^t(t_1, y_{0_t})) \ldots) \end{array}$$

and for every point $y_{0_t} \in \pi_{X_t}^{-1}(0_t)$ one has

$$H^t_{*(t_1,\dots,t_{l-1},0,y_{0_t})}(E_i) \;=\; H_{t*(t_1,\dots,t_{l-1},y_{0_t})}(E_i) \;=\; u_i^t(y_{l-1,t})\,, \qquad \forall\, i=1,\dots,l-1\,.$$

Let $\mathcal{H}^t = \{M_{y_{0_t}} := H^t(\mathbb{R}^l \times \{y_{0_t}\})\}_{y_{0_t}\in\pi_X^{-1}(0_t)}$ be the foliation defined by H^t. Then the frame field $(w_1^t,\dots,w_l^t)$ defined by

$$w_i^t(y) = H^t_{*(t_1,\dots,t_l,y_{0_t})}(E_i)\,, \qquad\qquad \forall\; i=1,\dots,l-1\,.$$

is the unique commuting (π_X,ρ_X)-controlled frame field tangent to $\mathcal{H}^t$, generating $T_y\mathcal{H}^t$ (Proposition 3, §5.2) and lifting $(E_1,\dots,E_l)$ on the leaves of $\mathcal{H}^t$ and it coincides with the frame field $(u_1^t,\dots,u_{l-1}^t,v_l)$ on every point $y_{l-1,t} = H_t(t_1,\dots,t_{l-1},y_{0_t}) \in Y_t$

For every $i,j=1,\dots,l-1$, since $[u_i^t,u_j^t]=0$, the flows $\psi^t_{i\,a}$, $\psi^t_{j\,b}$ of u_i^t, u_j^tcommute for all times $a,b\in\mathbb{R}$, and so using the relation $\psi^t_{i\,a}\psi^t_{j\,b} = \psi^t_{j\,b}\psi^t_{i\,a}$ before differentiating (see $[\mathbf{Mu}]_1$) for every $t\in[-1,1]$ and $y = H^t(t_1,\dots,t_l,y_{0_t})\in Y$ we obtain the equalities :

$$\begin{cases} w_l^t(y) = v_l(y) \\[2ex] w_i^t(y) := H^t_{*(t_1,\dots,t_l,y_{0_t})}(E_i) = \phi_{l\,t_l * y_{l-1}}(u_{i-1}^t(y_{l-1})) \quad \text{with the notation in §5.2 for } y_{l-1}. \end{cases}$$

By continuity of each $H^t_{*(t_1,\dots,t_{l-1},0,y_{0_t})}$ on $X_t\times\pi_{X_tY_t}^{-1}(0_t)$, and since

$$H^t_{*(t_1,\dots,t_{l-1},0,y_{0t})}(E_i) \;=\; w_i^t(y) \;=\; u_i^t(y)$$

for every $\epsilon>0$ there exists an open neighbourhood W_t of $Y_t = \pi_{X_tY_t}^{-1}(X_t)$ such that

$$\big|\big|w_i^t(y) - u_i^t(y)\big|\big| < \epsilon\,, \qquad \text{i.e.} \quad T_y\mathcal{H}^t \text{ is } \epsilon\text{-close to } \mathcal{D}_X \qquad \text{for every } y\in W_t\,,$$

and moreover

$$\bigcup_{t\in[-1,1]} W_t \;\supseteq\; \bigcup_{t\in[-1,1]} \pi_{X_tY_t}^{-1}(X_t) \;=\; \pi_{X_tY_t}^{-1}\Big(\bigcup_{t\in[-1,1]} X_t\Big) \;\supseteq\; \pi_{X_tY_t}^{-1}([-1,1]^l) \;\equiv\; Y\,.$$

Then the family $\mathcal{S}^{n+1} := \{V_t := W_t\cap A_{n+1}\}_{t\in[-1,1]}$ is an open covering of the compact subset $A_{n+1} = \cup_{t\in[-1,1]}A_{n+1,t}$ of Y and hence there exists a finite subfamily $\{V_{t_j}\}_j$ covering A_{n+1}.

In a similar way as in the first part of the proof (before step 1), for $\epsilon = \frac{1}{n+1}$ there exists $\delta>0$ and $s_n\in\mathbb{N}^*$ with $\delta := \frac{1}{s_n}$ such that we can obtain every V_{t_j} of the form :

$$V_{t_j} \;\supseteq\; H\Big([-1,1]^{l-1}\times Q_j\times\pi_{X_{t_j}}^{-1}(0_{t_j})\Big)\cap A_{n+1}$$

where $Q_j := [j\delta,(j+1)\delta]$ for every $j\in J_n := \{-s_n\dots,0,\dots,s_{n-1}\}$ and $\cup_{J\in J_n}Q_j = [-1,1]$.

Following exactly the same construction as in step 3, the foliations $\mathcal{H}^j := \mathcal{H}^{t_j}_{|V_{t_j}}$ with $j\in J_n$, induced by each $\mathcal{H}^{t_j}$ on V_{t_j}, glue together in a unique foliation

$$\mathcal{F}^{n+1} \;:=\; \mathcal{H}^{-s_n}\vee\dots\vee\mathcal{H}^{-1}\vee\;\mathcal{H}^0\vee\mathcal{H}^1\;\vee\;\dots\;\vee\;\mathcal{H}^{s_{n-1}}$$

of an open set

$$\bigcup_{J\in J_n} V_{t_j} \;\supseteq\; \bigcup_{J\in J_n} H\Big([-1,1]^{l-1}\times(\bigcup_{J\in J_n} Q_j)\times\pi^{-1}_{X_{t_j}}(0_{t_j})\Big)\cap A_{n+1} \;=$$

$$=\; H\Big([-1,1]^{l}\times\pi_X^{-1}(0_{t_j})\Big)\cap A_{n+1} \;=\; \pi_X^{-1}\big([-1,1]^l\big)\cap A_{n+1}\,.$$

Moreover as in Remark 6, since each $\mathcal{H}^j=\mathcal{H}^{t_j}_{|V_{t_j}}$ is $\frac{1}{n}$-close to $\mathcal{D}_{XY}$ on V_{t_j}, the global foliation $\mathcal{F}^{n+1}$ of A_{n+1} is $\frac{1}{n}$-close to $\mathcal{D}_{XY}$ too on the open set $\cup_{j\in J_n}V_{t_j}$.

We obtain thus for every $n\in\mathbb{N}^*$ a foliation $\mathcal{F}^{n+1}$ which is $\frac{1}{n}$-close to $\mathcal{D}_{XY}$.

At this point with formally the same proof as in step 4 one obtains an "increasing" sequence $\{\mathcal{K}_n\}_{n\geq 1}$ of foliations

$$\mathcal{K}_n := \Big(\big((\mathcal{F}^1\vee\mathcal{F}^2)\vee\ldots\vee\mathcal{F}^n\Big)\vee\mathcal{F}^{n+1} \quad \text{of the set} \quad A_1\cup\ldots\cup A_{n+2}\;\subseteq\;\rho^{-1}_{XY}([\tfrac{1}{n+3}\,,\,1])$$

with $\mathcal{K}_n$ coinciding with $\mathcal{K}_{n-1}$ on $\rho^{-1}_{XY}([\frac{1}{n}\,,\,1])$ and $\frac{1}{n-1}$-close to $\mathcal{D}_X$ on A_n.

In this way the restrictions $\mathcal{K}'_n := \mathcal{K}_{n|\rho^{-1}_{XY}([\frac{1}{n}\,,\,1])}$ define an increasing sequence of foliations with each $\mathcal{K}'_n$ $\frac{1}{n-1}$-close to $\mathcal{D}_X$ on A_n.

Finally, by taking on the whole of $W=\pi^{-1}_{XY}([-1,1]^l\cap\rho^{-1}_{XY}(]0,1])$ the foliation union $\mathcal{K}'=\cup_{n=1}^{\infty}\mathcal{K}'_n$, by (c)-regularity $[\mathbf{MT}]_2$, the canonical distribution is continuous on X and we conclude that :

$$\lim_{\substack{y\to x\in X\\ y\in Y}} T_y\mathcal{K}' \;=\; \lim_{\substack{y\to x\in X\\ y\in Y}} \mathcal{D}_{XY}(y) \;=\; T_xX\,. \qquad \square$$

Corollary 3. *With the hypotheses of Theorem 4, the open l-foliated neighbourhood W of $\pi_X^{-1}(U_{x_0})\cap T_X(1)$ may be chosen of type $\pi_X^{-1}(U')\cap T_X(1)$, where U' is the maximal domain of a chart near x_0 of X as a submanifold of $\mathbb{R}^n$.*

Proof. Let U' be a maximal domain of a chart $\phi: U'\xrightarrow{\equiv}\mathbb{R}^l\times\{0^k\}$ near $x_0\in X$.

By the Thom-Mather Isotopy Theorem there exists a topological trivialisation of $\mathcal{X}$ near x_0 :

$$H=H_{x_0} \quad : \quad \pi^{-1}_{XY}(\{x_0\})\times U' \;\equiv\; \pi^{-1}_{XY}(\{x_0\})\times\mathbb{R}^l \;\longrightarrow\; \pi^{-1}_{XY}(U')$$

having its values on the whole of $\pi^{-1}_{XY}(U')$.

Let us consider Theorem 4 proved for such a maximally defined map $H=H_{x_0}$.

In Theorem 4 we proved that starting from the compact set $[-1,1]^l\times\{0^k\}\subseteq\mathbb{R}^l\times\{0^k\}$, there exists a bounded neighbourhood W of 0^n in $\mathbb{R}^n$ containing the relatively compact set $\widetilde{U}=\pi^{-1}_{XY}([-1,1]^l\times\{0^k\})\cap T_X(1)$, and there exists a foliation $\mathcal{U}$ of $\widetilde{U}$ which is (a)-regular on all points of $[-1,1]^l\times\{0^k\}$, i.e. satisfying :

$$\lim_{y\to x} T_y\mathcal{U}=T_xX \qquad \text{for every} \quad x\in[-1,1]^l\times\{0^k\}\,.$$

Following the proof of Theorem 4 it is clear that the compact cube $[-1,1]^l\times\{0^k\}$ can be replaced by the bigger cube $U_n:=[-n,n]^l\times\{0^k\}$. The same proof holds allowing us to find an l-foliation $\mathcal{U}_n$ of an open bounded neighbourhood W'_n of the relatively compact set $\widetilde{U}_n := \pi_{XY}^{-1}([-n,n]^l \times \{0^k\}) \cap T_X(1)$ which is (a)-regular on all points of U_n.

At this point the proof follows using Zorn's Lemma to give the existence of a maximal element of the set of all (a)-regular l-foliations each of whose domains contains a set of the sequance $\pi_{XY}^{-1}([-n,n]^l \times \{0^k\}) \cap T_X(1)$ with respect to an appropriate partial order relation.

However, we give a constructive proof as follows.

We prove by induction that the sequence of these (a)-regular foliations $\{\mathcal{U}_n\}_n$ may be modified to a new sequence of (a)-regular foliations $\{\mathcal{U}'_n\}_n$ in which each foliation $\mathcal{U}'_{n+1}$ defined on $\widetilde{U}_{n+1}$ is an extension of $\mathcal{U}_{n|\widetilde{U}_{n-1}}$.

Let $\mathcal{U}'_n$ be a foliation on $\widetilde{U}_n = \pi_X^{-1}([-n,n]^l \times \{0^k\}) \cap T_X(1)$ extending the restriction of $\mathcal{U}'_{n-1}$ to $\widetilde{U}_{n-2}$ and (a)-regular on all points of U_n.

Via a sequence of two gluings, using the same techniques as in the proofs of Step 2 and Step 3 in Theorem 4, we glue $\mathcal{U}_n$ and the restriction $\mathcal{U}_{n+1|U_{n+1}-U_{n-1}}$ and define a new l-foliation :

$$\mathcal{U}'_{n+1} \quad \text{which} \ : \ \begin{cases} \text{coincides with } \mathcal{U}'(n) & \text{on} \ \ \widetilde{U}'_{n-1} \ ; \\ \text{coincides with } \mathcal{U}_{n+1} & \text{on} \ \ \widetilde{U}_{n+1} - \widetilde{U}_n \ ; \\ \text{is } (a)\text{-regular} & \text{on} \ \ U_{n+1}. \end{cases}$$

We have then an increasing sequence of (a)-regular foliations : $\mathcal{U}'_{n|\widetilde{U}_{n-1}}$ whose union is defined on the set

$$\bigcup_n \widetilde{U}_{n-1} \ = \ \pi_{XY}^{-1}\Big(\bigcup_n [-n,n]^l \times \{0^k\}\Big) \cap T_{XY}(1) = \pi_{XY}^{-1}\big(\mathbb{R}^l \times \{0^k\}\big) \cap T_{XY}(1)$$

and which is (a)-regular on all points of :

$$\bigcup_{n=1}^{+\infty} U_{n-1} \ = \ \bigcup_n [-n,n]^l \times \{0^k\} \ = \ \mathbb{R}^l \times \{0^k\} \ \equiv \ U' \, . \qquad \square$$

6. Local regular open book structures.

In this section, we prove Theorem 5 and Theorem 6 in which we construct a local open book structure for Bekka (c)-regular and Whitney (b)-regular stratifications for every stratum X with $depth_\Sigma(X) = 1$.

These partial results (since $depth_\Sigma(X) = 1$) will play an important role in the proof of our main Theorem 7 of section 7 and will be extended to the general case of an arbitraty $depth_\Sigma(X)$ as corollaries of Theorem 7.

Definition 8. Let $\mathcal{X} = (A, \Sigma)$ be a smooth (a)- or (c)- or (b)-regular stratification in $\mathbb{R}^n$, $X \in \Sigma$ and $x_0 \in X$.

One says that $\mathcal{X}$ admits a *local open book structure at (or near)* x_0 if there exists a system of control data $\mathcal{F} = \{(\pi_X, \rho_X, T_X)\}_{X\in\Sigma}$, a neighbourhood U_{x_0} of x_0 in X and $\epsilon > 0$ such that the stratified space $\big(\pi_X^{-1}(U_{x_0}) - U_{x_0}\big) \cap T_X(\epsilon)$ has a stratified foliation

$$\mathcal{W}_{x_0} = \big\{W_{y_0} \ \mid \ y_0 \in (\pi_X^{-1}(x_0) - \{x_0\}) \cap S_X(\epsilon)\big\} \, , \qquad\qquad y_0 \in W_{y_0}$$

such that for every stratum $Y > X$ and $y_0 \in Y$,

i) W_{y_0} is a C^∞-submanifold of $T_{XY}(\epsilon)$ containing y_0 ;
ii) $U_{x_0} \subseteq \overline{W_{y_0}}$ (frontier condition) ;
iii) the restriction $(\pi_{XY}, \rho_{XY})_{|W_{y_0}} : W_{y_0} \longrightarrow U_{x_0} \times]0, \epsilon[$ is a C^∞-diffeomorphism.

If these conditions hold, each stratification $W_{y_0} \sqcup U_{x_0}$ is called a *local page at* x_0 *in* Y.

The local open book structure $\mathcal{W}_{x_0}$ is called (a)- *or* (c)- *or* (b)*-regular* respectively if moreover :

iv) every pair of strata $U_{x_0} < W_{y_0}$ is (a)- or (c)- or (b)–regular.

If such conditions are satisfied we also say that $\mathcal{W}$ *is a local* (a)- *or* (c)- *or* (b)*-regular open book structure over* U (omitting x_0).

In 1976 **[Go]**$_1$ Goresky introduced the following very useful notion :

Definition 9. Let $\mathcal{X} = (A, \Sigma)$ be an abstract stratified set, a family of maps

$$\left\{ r_X^\epsilon : T_X(1) - X \to S_X(\epsilon) \right\}_{X \in \Sigma\,,\, \epsilon \in]0,1[},$$

is said to be *a family of lines for* $\mathcal{X}$ *(with respect to a given system of control data)* $\left\{ (T_X, \pi_X, \rho_X) \right\}$ if for every pair of strata $X < Y$, the following properties hold :

1) every restriction $r_{XY}^\epsilon := r_{X|Y}^\epsilon : T_{XY} \longrightarrow S_{XY}(\epsilon)$ of r_X^ϵ is a C^1-map ;
2) $\pi_X \circ r_X^\epsilon = \pi_X$;
3) $r_X^{\epsilon'} \circ r_X^\epsilon = r_X^{\epsilon'}$;
4) $\pi_X \circ r_Y^\epsilon = \pi_X$;
5) $\rho_Y \circ r_X^\epsilon = \rho_Y$;
6) $\rho_X \circ r_Y^\epsilon = \rho_X$;
7) $r_Y^{\epsilon'} \circ r_X^\epsilon = r_X^\epsilon \circ r_Y^{\epsilon'}$.

In order to obtain his important theorem of triangulation of abstract stratified sets, **[Go]**$_3$ Goresky proved that every abstract stratified set $\mathcal{X}$ admits a family of lines.

Since (c)-regular **[Be]** and a fortiori (b)-regular **[Ma]** stratifications admit structures of abstract stratified sets a family of lines exists for them.

We can now prove the following :

Theorem 5. *Let* $\mathcal{X} = (A, \Sigma)$ *be a closed smooth Bekka* (c)*-regular stratified subset of* $\mathbb{R}^n$. *Then for every stratum* X *of* $depth_\Sigma(X) = 1$, *each pair of strata* $X < Y$ *admits a local* (c)*-regular open book structure near every* $x_0 \in X$.

Proof. Since the pair of strata $X < Y$ is (c)-regular, by Theorem 3 there exists a neighbourhood U_{x_0} in X (which in a local analysis we identify with $\mathbb{R}^l \times \{0\} \subseteq \mathbb{R}^n$), and a local (a)-regular foliation $\mathcal{H}_{x_0} = \left\{ M_{y_0} := H(\mathbb{R}^l \times \{y_0\}) \right\}_{y_0 \in \pi_{XY}^{-1}(x_0)}$ corresponding to a stratified local topological trivialization :

$$\begin{array}{rccc} H: & \mathbb{R}^l \times \pi_X^{-1}(x_0) & \longrightarrow & \pi_X^{-1}(\mathbb{R}^l \times \{0\}) \subseteq \mathbb{R}^n \\ & (t_1, \ldots, t_l, y_0) & \longmapsto & y := \phi_l(t_l, \ldots, \phi_1(t_1, y_0) \ldots) \end{array}$$

where $\{(\pi_{XY}, \rho_{XY}) : T_{XY} \to X \times]0,1[\}$ is the C^∞-submersion of a system of control data.

As $X \sqcup Y$ is (c)-regular it is an abstract stratified set **[Be]** and hence it admits a family of lines $\left\{ r_X^\epsilon : T_X(1) - X \to S_X(\epsilon) \right\}_{\epsilon \in]0,1[}$ **[Go]**$_3$.

For every y_0 in the link $L(x_0,\epsilon) := S_X(\epsilon) \cap \pi_{XY}^{-1}(x_0)$ we consider the C^∞-arc

$$\gamma_{y_0} \ : \ ]0,\epsilon[\longrightarrow \pi_{XY}^{-1}(x_0)\,, \qquad \gamma_{y_0}(s) = r_X^s(y_0)$$

which is a C^∞-diffeomorphism on its image and we define the foliation

$$\mathcal{L}_{x_0} := \Big\{ L_{y_0} := \gamma_{y_0}(]0,\epsilon[) \Big\}_{y_0 \in L(x_0,\epsilon)}$$

by 1-dimensional arcs of the fiber $\pi_{XY}^{-1}(x_0) \cap T_{XY}(\epsilon)$ parametrized in the link $L(x_0,\epsilon)$.

Since $\gamma_{y_0}(s) \subseteq S_X(s) = \rho_X^{-1}(s)$ and $\rho_X^{-1}(0) = X$, one has $\lim_{s\to 0}\gamma_{y_0}(s) = x_0$. Hence each line L_{y_0} satisfies : $\{x_0\} \subseteq \overline{L_{y_0}}$.

For every $y_0 \in L(x_0,\epsilon)$, setting $W_{y_0} := H(\mathbb{R}^l \times L_{y_0})$ the family

$$\mathcal{W}_{y_0} \ := \ \big\{W_{y_0}\big\}_{y_0 \in L(x_0,\epsilon)}$$

defines a foliation satisfying the local open book properties near x_0.

In fact, since H is a homeomorphism it is easy to see that $U_{x_0} \subseteq \overline{W_{y_0}}$ for every y_0.

Since H is a diffeomorphism on strata, every leaf $W_{y_0} = H(\mathbb{R}^l \times L_{y_0})$ is a C^∞-submanifold of $T_{XY}(\epsilon)$ of dimension $(l+1)$.

Moreover $(\pi_{XY},\rho_{XY}) : T_{XY} \to X \times]0,1[$ being a C^∞-submersion, its restriction to each leaf $(\pi_{XY},\rho_{XY})_{|W_{y_0}} : W_{y_0} \to X \times]0,\epsilon[$ is a C^∞-diffeomorphism.

Finally, for every $x = (t_1,\dots,t_l) \in U_{x_0} \equiv \mathbb{R}^l$ we have that, for every $y \in W_{y_0}$, there exists $s \in]0,\epsilon[$ such that $y = H((t_1,\dots,t_l,\gamma_{y_0}(s))$ and so that

$$W_{y_0} \ = \ H(\mathbb{R}^l \times L_{y_0}) \ \supseteq \ H\big(\mathbb{R}^l \times \gamma_{y_0}(s)\big) \ = \ M_{\gamma_{y_0}(s)}$$

and hence by (a)-regularity of the foliation $\mathcal{H}_{x_0}$ one finds :

$$\lim_{y\to x} T_y W_{y_0} \ \supseteq \ \lim_{y \to x} T_y M_{\gamma_{y_0}(s)} \ \supseteq \ T_x X$$

which proves (a)-regularity of the pair of strata $U_{x_0} < W_{y_0}$ at every $x \in U_{x_0}$.

Finally by considering the distance function $\rho_{U_{x_0}W_{y_0}}$, the restriction of ρ_{XY}, each level hypersurface satisfies :

$$\rho^{-1}_{U_{x_0}W_{y_0}}(\epsilon) \ = \ \rho_{XY}^{-1}(\epsilon) \cap W_{y_0} \ = \ M_{y_0}$$

and hence (c)-regularity of $U_{x_0} < W_{y_0}$ follows by (a)-regularity of the foliation $\mathcal{H}_{x_0} = \{M_y\}_y$:

$$\lim_{y\to x} \rho^{-1}_{U_{x_0}W_{y_0}}(\epsilon) \supseteq \lim_{y\to x} T_y M_y = T_x X. \quad \square$$

For a (b)-regular stratification, with the aim of proving a corresponding (b)-regular open book structure near $x_0 \in X$, we cannot use an arbitrary Goresky family of lines because these lines are not necessarily (b)-regular over x_0. Fortunately this result holds if the lines are the integral curves of the gradient of the distance function ρ_X.

In **[MT]**$_5$ we construct a family of lines $\big\{r_X^\epsilon\big\}_{X\in\Sigma\,,\,\epsilon\in]0,1[}$, with this desirable property; however for the simpler case $depth_\Sigma(X) = 1$, in which only two strata $X < Y$ and one map (π_X, ρ_X) occur we do not need all compatibility conditions of the family of lines.

In Theorem 6 below we will not use the result obtained in $[\mathbf{MT}]_5$.

Theorem 6. *Let $\mathcal{X} = (A, \Sigma)$ be a closed smooth Whitney (b)-regular stratified subset of $\mathbb{R}^n$. For every stratum X of $depth_\Sigma(X) = 1$, each pair of strata $X < Y$ admits a local (b)-regular open book structure near every $x_0 \in X$.*

Proof. The pair of strata $X < Y$ being (b)-regular, it is (c)-regular too **[Be], [Tr]**$_1$, hence by Theorem 3 there exists a neighbourhood U_{x_0} in X, which in a local analysis we identify with $\mathbb{R}^l \times \{0\} \subseteq \mathbb{R}^n$, and there exists a local (a)-regular foliation $\mathcal{H}_{x_0} = \big\{M_{y_0} := H(\mathbb{R}^l \times \{y_0\})\big\}_{y_0 \in \pi_{XY}^{-1}(x_0)}$ obtained from the stratified local topological trivialization :

$$H: \quad \mathbb{R}^l \times \pi_X^{-1}(x_0) \quad \longrightarrow \qquad \pi_X^{-1}(\mathbb{R}^l \times \{0\}) \subseteq \mathbb{R}^n$$

$$(t_1, \ldots, t_l, y_0) \quad \longmapsto \quad y := \phi_l(t_l, \ldots, \phi_1(t_1, y_0) \ldots)$$

where $\{(\pi_{XY}, \rho_{XY}) : T_{XY} \to X \times]0,1[\}$ is the C^∞-submersion of a system of control data.

Let us consider the distance function $\rho_{XY} : T_{XY}(\epsilon) \to X$ and on $T_{XY}(\epsilon)$, the vector field $v(y) := \nabla\rho_{XY}(y)$ and the integral flow $\phi : \mathbb{R} \times T_{XY}(\epsilon) \to T_{XY}(\epsilon)$ of v.

Since $\rho_{XY} : T_{XY}(\epsilon) \to X$ is a submersion, $\nabla\rho_{XY}(y) \neq 0 \ \forall\, y \in T_{XY}(\epsilon)$.

For every y in the link $\pi_{XY}^{-1}(x) \cap T_{XY}(\epsilon)$ we consider the C^∞-arc

$$\gamma_y \quad : \quad]-\infty, 0[\longrightarrow \pi_{XY}^{-1}(x)\,, \qquad \gamma_y(s) = \phi(s, y)$$

which is a C^∞-diffeomorphism on its image.

For every $x \in X$ we define the foliation

$$\mathcal{L}_x := \Big\{L_{y_0} := \gamma_{y_0}(]-\infty, 0[)\Big\}_{y_0 \in L(x,\epsilon)}$$

of the fiber $\pi_{XY}^{-1}(x) \cap T_{XY}(\epsilon)$ by arcs parametrized in the link $L(x, \epsilon) := \pi_{XY}^{-1}(x_0) \cap S_{XY}(\epsilon)$.

We write $L_y := L_{y_0}$ if $y = \gamma_{y_0}(s)$ is in the same trajectory $\gamma_{y_0}(]-\infty, 0[)$ as y_0.

Moreover, for every $s \in]-\infty, 0[$, by $\gamma_y(s) \subseteq S_X(s) = \rho_X^{-1}(s)$ and $\rho_X^{-1}(0) = X$, one has $\lim_{s \to -\infty} \gamma_y(s) = x$ and hence each line L_y satisfies : $\{x\} \subseteq \overline{L_y}$, with $x = \pi_{XY}(y)$.

For every $y \in T_{XY}(\epsilon)$, setting $W_y := H(\mathbb{R}^l \times L_y)$ the family

$$\mathcal{W}_y \ := \ \Big\{W_y \ := \ H(\mathbb{R}^l \times L_y)\Big\}_{y \in T_{XY}(\epsilon)}$$

defines a foliation for which in the same way as for the (c)-regular case one proves that it satisfies the local (a)-regular open book properties near x_0.

Recall now $[\mathbf{Tr}]_1$ the following two useful characterizations of (b)-regularity at $x \in X < Y$ for two strata $X < Y$ of a stratification in $\mathbb{R}^n$:

i) $X < Y$ is (b)-regular at $x \in X$ if and only it is (a)- and (b^π)-regular with respect to each C^∞-projection $\pi_X : T_X \to X$.

ii) $X < Y$ (b)-regular at $x \in X$ implies $[\mathbf{Ma}]_{1,2}$ that in local coordinates there exist control data (π_X, ρ_X) where π is the canonical projection $\pi(t_1, \ldots, t_n) = (t_1, \ldots, t_l, 0^{n-l})$ and ρ the standard distance from $\mathbb{R}^l \times 0^{n-l}$, $\rho(t_1, \ldots, t_n) = \sum_{i=l+1}^n t_i^2$.

By *i)* it remains to prove that $U_{x_0} < W_{y_0}$ is (b^π)-regular at each point $x \in U_{x_0}$.

Let us fix $x \in X$. To simply notations we identify $T_{XY}(\epsilon)$ and Y.

By definition of (b^π)-regularity at $x \in X$, we must prove that for every sequence $\{y_n\}_n \subseteq W_{y_0}$ such that $\lim_n y_n = x$ and both limits below exist in the appropriate Grassmann manifolds, i.e.

$$\lim_n T_{y_n} W_{y_0} = \sigma \in \mathbb{G}_n^{l+1} \quad \text{and} \quad \lim_n \overline{y_n \pi_X(y_n)} = L \in \mathbb{G}_n^1, \quad \text{then} \quad \sigma \supseteq L.$$

The Grassmann manifold $\mathbb{G}_n^{\dim Y}$ being compact, taking a subsequence if necessary we can suppose that $\lim_n T_{y_n} Y = \tau \in \mathbb{G}_n^{l+1}$.

By hypothesis $X < Y$ is (b)-regular and hence (b^π)-regular at $x \in X$ so that $\tau \supseteq L$.

Moreover by *ii)* we can assume that $\pi_X = \pi : \mathbb{R}^n \to \mathbb{R}^l \times 0^{n-l}$ and ρ_X is the standard distance $\rho(t_1, \ldots, t_n) = \sum_{i=l+1}^n t_i^2$, so that $\nabla \rho_X(y) = 2(y - \pi_X(y))$ and they generate the same vector space $[\nabla \rho_X(y)] = [y - \pi_X(y)]$.

For every $n \in \mathbb{N}$, let u_n be the unit vector $u_n := \frac{y_n - x_n}{||y_n - x_n||}$ where $x_n = \pi_X(y_n)$.

For every vector subspace $V \subseteq \mathbb{R}^n$, let $p_V : \mathbb{R}^n \to V$ be the orthogonal projection on V and let us consider the "distance" function defined by (**[Ve]**, $\mathbf{[Mu]}_2$ §4.2) :

$$\begin{cases} \delta(u, V) \;=\; \inf_{v \in V} \; ||\, u - v \,|| \;=\; ||\, u - p_V(u)|| & \text{for every } u \in \mathbb{R}^n \\ \text{and} & \\ \delta(U, V) \;=\; \sup_{u \in U, ||\, u\,|| = 1} \; ||\, u - p_V(u)\,|| & \text{for every subspace } U \subseteq \mathbb{R}^n\,. \end{cases}$$

Then by (b^π)-regularity of $X < Y$ it follows that :

$$\tau \supseteq L \quad \Longrightarrow \quad \lim_n \, [u_n] \subseteq \lim_n \; T_{y_n} Y \quad \Longrightarrow \quad \lim_n \; \delta([u_n], T_{y_n} Y) = 0\,.$$

Since ρ_{XY} is the restriction $\rho_{X|Y}$ of ρ_X to Y, every vector $\nabla \rho_{XY}(y_n)$ is the orthogonal projection $p_{T_{y_n} Y}(\nabla \rho_X(y_n))$ on $T_{y_n} Y$ of the vector $\nabla \rho_X(y_n)$ and we have :

$$T_{y_n} L_{y_n} = [\nabla \rho_{XY}(y_n)] = p_{T_{y_n} Y}(\nabla \rho_X(y_n)) = p_{T_{y_n} Y}([y_n - x_n]) = p_{T_{y_n} Y}([u_n])$$

by which, u_n being a unit vector of $[u_n]$, one finds that :

$$\delta([u_n], T_{y_n} L_{y_n}) = \delta([u_n], p_{T_{y_n} Y}([u_n])) = ||\; u_n - p_{T_{y_n} Y}(u_n)\,|| = \delta([u_n], T_{y_n} Y)\,.$$

On the other hand $L_{y_n} \subseteq W_{y_0} \subseteq Y$, so $T_{y_n} L_{y_n} \subseteq T_{y_n} W_{y_0} \subseteq T_{y_n} Y$, and hence :

$$0 \;\leq\; \lim_{y_n \to x} \delta([u_n], T_{y_n} W_{y_0}) \;\leq\; \lim_{y_n \to x} \delta([u_n], T_{y_n} L_{y_n}) \;=\; \lim_{y_n \to x} \delta([u_n], T_{y_n} Y) \;=\; 0\,.$$

We deduce that $\lim_{y_n \to x} \delta([u_n], T_{y_n} W_{y_0}) = 0$ and this implies :

$$L = \lim_{y_n \to x} \overline{y_n \pi_X(y_n)} = \lim_{y_n \to x} [u_n] \subseteq \lim_{y_n \to x} T_{y_n} W_{y_0} = \sigma$$

which proves that $U_{x_0} \equiv \mathbb{R}^l \times 0^{n-l} < \mathcal{W}_{y_0}$ is (b^π)-regular at x, for every $x \in U_{x_0}$. $\square$

7. Proof of the smooth Whitney fibering conjecture in the general case.

In this section we give our main results. First we use the local open book structure of section 6 to prove the conclusions of the smooth Whitney fibering conjecture for a stratum X of a (c)-regular stratification $\mathcal{X} = (A, \Sigma)$ having arbitrary depth (Theorem 7) and then we use Theorem 7 to extend Theorems 5 and 6 of section 6 to a stratum of arbitrary depth (Theorem 8).

The definition below will be useful in the proof of Theorem 7. A similar notion (Σ-chart) was introduced in **[Fer]**.

Definition 10. Let $\mathcal{X} = (A, \Sigma)$ be an abstract stratified set with a fixed system of control data $\mathcal{T} = \{(T_X, \pi_X, \rho_X)\}_{X\in\Sigma}$.

Let $X^l < Y^k$ be two adjacent strata of $\mathcal{X}$, U the domain of a chart $\varphi : U \subseteq X \to \mathbb{R}^l$, and $(u_1, \dots, u_l)$ the frame field defined by $u_i := \varphi_*^{-1}(E_i)$ and $\mathbb{R}^k_+ := \mathbb{R}^{k-1}\times]0, +\infty[$.

We call *conical chart of Y over U* a chart $\tilde\varphi : \pi_{XY}^{-1}(U) \cap T_{XY}(\epsilon) \to \mathbb{R}^k_+$ of Y such that:

1) $\varphi \circ \pi_{XY} = p \circ \tilde\varphi$ where $p : \mathbb{R}^k \to \mathbb{R}^l$ is the canonical projection ;

2) $\forall\, \epsilon' \in]0, \epsilon[$ the restriction $\tilde\varphi_{\epsilon'} : \pi_{XY}^{-1}(U) \cap S_{XY}(\epsilon') \to \mathbb{R}^{k-1}$ of $\tilde\varphi$ is a chart of $S_{XY}(\epsilon')$;

3) $\tilde\varphi$ extends to the stratified homeomorphism $\varphi \sqcup \tilde\varphi : U \cup Y \to \mathbb{R}^l \times \{0^{k-l}\} \sqcup \mathbb{R}^k_+$;

Example 1. Let H be the topological trivialization of the projection $\pi_{XY} : T_{XY} \to X$:

$$\begin{array}{rcl} H = H_{x_0} \ : \ U \times \pi_{XY}^{-1}(x_0) \cong \mathbb{R}^l \times \pi_{XY}^{-1}(x_0) & \longrightarrow & \pi_{XY}^{-1}(U) \subseteq \mathbb{R}^n \\ (t_1, \dots, t_l, y_0) & \longmapsto & \phi_l(t_l, \dots, \phi_1(t_1, y_0)..) \end{array}$$

where $\forall i \le l$, ϕ_i is the flow of the vector field v_i which is the (π, ρ)-controlled lifting of u_i.

If

$$h : V \subseteq S_{XY}(\epsilon) \longrightarrow \mathbb{R}^{k-l-1} \quad \text{is a chart of the link} \quad L_{XY}(x_0, \epsilon) := \pi_{XY}^{-1}(x_0) \cap S_{XY}(\epsilon)$$

and we consider the families of smooth arcs as in §6 :

$$\big\{\gamma_{y_0} :]0, 1] \to \pi_{XY}^{-1}(x_0)\big\}_{y_0\in V} \quad \text{whose images are the lines} \quad \big\{L_{y_0} := \gamma_{y_0}(]0, 1])\big\}_{y_0\in V}$$

then the union $V' = \sqcup_{y_0\in V} L_{y_0}$ is the domain of a conical chart of $\pi_{XY}^{-1}(x_0)$ and the disjoint union of wings

$$U' = H(U \times V') \ := \ \sqcup_{y_0\in V} H(U \times L_{y_0})$$

is the domain of a conical chart $\tilde\varphi$ of Y over U defined by :

$$\begin{array}{rcl} \tilde\varphi \ : \ U' := H(U \times V') & \longrightarrow & U \times]0, \epsilon[\times \mathbb{R}^{k-l-1} \\ y = H(t_1, \dots, t_l, y_{0,t}) & \longmapsto & \tilde\varphi(y) := \big(\pi_{XY}(y), \rho_{XY}(y), h(y_0)\big) \end{array}$$

(where $y_{0,t} := \gamma_{y_0}(t)$) which satisfies :

$$\tilde\varphi_{*y}(v_i(y)) = \Big(\pi_{XY*y}(v_i(y)), \rho_{XY*y}(v_i(y)), h_{*y_0}(v_i(y_0))\Big) = (u_i(x), 0, 0^{k-l-1}) = (u_i(x), 0^{k-l}). \quad \square$$

Remark 7. With the same notations as in Example 1 one has :

i) If $(q_1, \dots, q_k)$ denotes the coordinate frame field induced by $\tilde\varphi \ : \ U' \to U\times]0, \epsilon[\times \mathbb{R}^{k-l-1}$, then for every $y = H(t_1, \dots, t_l, y_{0,t}) \in U'$ we have :

$$q_i(y) = \begin{cases} \tilde\varphi_{*y}^{-1}(u_i(x), 0^{k-l}) \ = \ v_i(y) & \text{for every } i = 1, \dots, l \\ \tilde\varphi_{*y}^{-1}(E_{l+1}) \ = \ (\gamma_{y_0})'(t) & \text{for } \quad i = l+1 \\ \tilde\varphi_{*y}^{-1}(E_i) \ = \ \gamma_{t*y}(h_{*y_0}^{-1}(E_i)) & \text{for every } i = l+1, \dots, k \ ; \end{cases}$$

ii) If $\mathcal{A} := \{h_i : V_i \to \mathbb{R}^{k-l-1}\}_{i\in I}$ is an atlas of $L_{XY}(x_0,\epsilon)$, then

$$\cup_i V_i = L_{XY}(x_0,\epsilon) \quad \Rightarrow \quad \cup_i V_i' = \pi_{XY}^{-1}(x_0) \quad \Rightarrow \quad \cup_i U_i' = \pi_{XY}^{-1}(U)\cap T_{XY}(\epsilon)\,,$$

and hence

$$\tilde{\mathcal{A}} := \{\,\tilde{\varphi}_i : U_i' \to U_i \times]0,\epsilon[\times \mathbb{R}^{k-l-1}\,\}_{i\in I} \qquad \text{is an atlas of} \quad \pi_{XY}^{-1}(U)\cap T_{XY}(\epsilon).$$

iii) If the foliation $\mathcal{H}_{x_0}$ is (a)-regular over U, the stratified homeomorphism $\varphi \sqcup \tilde{\varphi} : U \sqcup U' \subseteq X \sqcup Y \to \mathbb{R}^l \times 0^{k-l}\} \sqcup \mathbb{R}^k_+$ is horizontally-C^1 over U for the stratification $U < U'$ (see $[\mathbf{MT}]_4$ or Section 8 for this definition). $\square$

The theorem below says that any Bekka (c)-regular stratification satisfies the conditions required by the smooth version of the Whitney fibering conjecture (see Corollary 4 below).

Theorem 7. *Let $\mathcal{X} = (A,\Sigma)$ be a smooth stratified Bekka (c)-regular subset of $\mathbb{R}^n$, X a stratum of X, $x_0 \in X$ and U a domain of a chart near x_0 of X.*

There exists a stratified foliation $\mathcal{F}_{x_0} = \{F_z\}_{z\in\pi_X^{-1}(x_0)}$ of the neighbourhood $W = \pi_X^{-1}(U)$ of x_0 in A whose leaves F_z are smooth l-manifolds diffeomorphic to $X\cap W$, such that for every stratum $X_j \geq X$, $X_j \cap W$ is a union of leaves and $\mathcal{F}_{x_0}$ satisfies:

$$(1): \qquad \lim_{\substack{z\to x_j\\ z\in A}} T_zF_z \;=\; T_{x_j}F_{x_j} \;\subseteq\; T_{x_j}X_j\,, \qquad \textit{for every} \quad x_j \in X_j\cap W\,,$$

and in particular for $X_j = X$ one has :

$$(2): \qquad \lim_{\substack{z\to x\\ z\in A}} T_zF_z \;=\; T_xF_x \;=\; T_xX\,, \qquad \textit{for every} \quad x\in X\cap W\,.$$

Proof. We prove the theorem by induction on $s = depth_\Sigma X$.

In Theorems 4 and 3 we proved the statement when $s = depth_\Sigma X = 1$; this provides the start of the induction.

Let X be an l-stratum of $\mathcal{X}$ having $depth_\Sigma X = s \geq 2$.

A tubular neighbourhood T_X of X is naturally stratified by strata $X^i_j \geq X$ (with $dim X^i_j = i$ and $j\in J_i$) and if T_X is sufficiently small every two strata of the same dimension X^i_j, $X^i_{j'}$ have disjoint tubular neighbourhoods $[\mathbf{Ma}]_{1,2}$. Then by interpreting all strata of the same dimension i as a unique (non connected) i-stratum of T_X we can suppose that T_X admits at most a unique stratum $X^i > X$ of each dimension $i > l = dim X$.

Hence it is sufficient to prove the theorem in the case where T_X has a unique chain of strata adjacent to X :

$$X = X_0 < X_1 < \cdots < X_{s-1} < X_s \qquad \text{with} \quad s\geq 2\,.$$

Let $U := U_{x_0} \subseteq X$ be a maximal domain of a chart $\varphi : U \to \mathbb{R}^l$ of X near $x_0\in X$ and set $Y := X_{s-1}$, $Z := X_s$, $k = \dim Y$.

The stratification $\mathcal{X}' = (A',\Sigma')$ obtained by removing from Σ all strata of dimension strictly bigger than $\dim X_{s-1}$ is obviously again (c)-regular with system of control data the family of restrictions $\{(\pi_{X|A'}\,,\,\rho_{X|A'})\}_{X\in\Sigma'}$, it has Y as biggest stratum and $depth_{\Sigma'}X = s-1$ so by inductive hypothesis the theorem holds for $\mathcal{X}'$ and there exists a stratified foliation $\mathcal{F}'_{x_0} := \{F'_y\}_{y\in\pi^{-1}_{X|A'}(x_0)}$ of the neighbourhood $W' := \pi_X^{-1}(U)\cap A'$ of x in A' satisfying the limit properties (1) and (2) for every $j = 0,\ldots,s-1$.

We denote by $T'_X := T_X \cap A'$, the tubular neighbourhood and by $S'_X(\epsilon) := S_X(\epsilon)\cap A'$ the ϵ-sphere of X in A' induced by the control data $\{(\pi_{X|A'}\,,\,\rho_{X|A'})\}_{X\in\Sigma'}$.

Let $(u_1,\ldots,u_l)$ be the frame field $u_i := \varphi_*^{-1}(E_i)$ induced by the chart φ and H the topological trivialization of the projection $\pi_X : T_X(1)\to X$,

$$H\ :\ U\times\pi_X^{-1}(x_0)\equiv\mathbb{R}^l\times\pi_X^{-1}(x_0)\ \longrightarrow\ \pi_{XY}^{-1}(U)\subseteq\mathbb{R}^n$$

$$(t_1,\ldots,t_l,z_0)\ \longmapsto\ \phi_l(t_l,\ldots,\phi_1(t_1,z_0)..)$$

where (as in §5.1) $\forall i = 1,\ldots,l$, ϕ_i is the stratified flow of the stratified vector field v_i which is the continuous controlled lifting in the stratified space $\pi_X^{-1}(U) = \sqcup_{X'\ge X}\,\pi_{XX'}^{-1}(U)$ of the coordinate vector field u_i defined on $U\subseteq X$.

According to $[\mathbf{MT}]_2$ (where the proof is given for the three strata case) for each vector field u on X a stratified continuous controlled vector field w on $T_X(1)$ is obtained inductively by the following steps:

i) one lifts u to a continuous (π_{XZ},ρ_{XZ})-controlled vector field u^{XZ} on $T_{XZ}(1)$ in A ;

ii) one lifts by induction u to a continuous (π,ρ)-controlled vector field $u^{XA'}$ on the stratified space $T'_X(1)$ in which $depth_{\Sigma'}X = s-1$ and having Y as maximum stratum ;

iii) one lifts the restriction $u^{XY} := u^{XA'}_{|T_{XY}(1)}$ to a continuous (π_{YZ},ρ_{YZ})-controlled vector field u^{YZ} on $T_{YZ}(\epsilon)\cap T_{XZ}(1)$ in A where

$$(*):\qquad T_{YZ}(\epsilon) := \Big\{z\in T_{XZ}(1)\ \big|\ ||u_{YZ}(z)-u_{XY}(y)|| < d(y,X)\ \Big\}\quad\text{with}\quad y=\pi_{YZ}(z)$$

($d :=$ usual distance of $\mathbb{R}^n$) and making a change of factor we rename it by $T_{YZ}(1)$;

iv) one glues u^{XZ} and u^{YZ} by a partition of unity subordinate to the open covering $\mathcal{O} := \{O_1\,,O_2\}$ of $T_X(1)$ where :

$$O_1 := T_{XZ}(1) - \overline{T_{YZ}(1/2)}\qquad\text{and}\qquad O_2\ :=\ T_{XZ}(1)\cap T_{YZ}(1)\,.$$

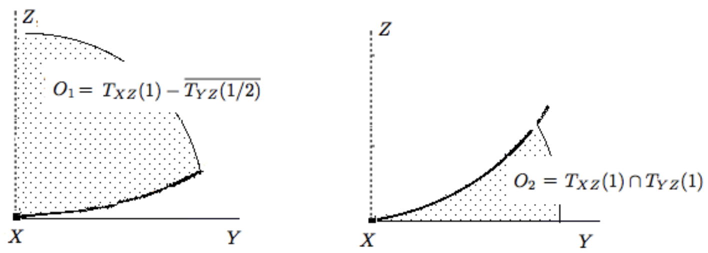

Figure 9

This gives a stratified vector field w on $T_X(1)$, a lifting of the vector field u on X, which is continuous and controlled with respect to all strata of $T_X(1)$.

We first apply this construction to obtain continuous controlled lifted vector fields $(v_1,\ldots,v_l)$ defining the trivialization H above and to obtain the induced l-foliation (not necessarily (a)-regular) :

$$\mathcal{H}_{x_0} = \{M_y^{x_0}\}_{y\in\pi_X^{-1}(x_0)}\ =\ \{M_y^{x_0}\}_{y\in\pi_X^{-1}(x_0)\cap A'}\ \bigsqcup\ \{M_z^{x_0}\}_{z\in\pi_{XZ}^{-1}(x_0)}\,.$$

Then we apply Theorems 3 and 4 to these two sub-foliations of $\mathcal{H}_{x_0}$ to construct two (a)-regular l-foliations, whose leaves (with a slight abuse of notation) we denote again by $M_y^{x_0}$ and $M_z^{x_0}$:

$$\begin{cases} \mathcal{H}_{x_0,Z} \; = \; \{M_z^{x_0}\}_{z\in\pi_{XZ}^{-1}(x_0)} & \text{of} \quad \pi_{XZ}^{-1}(U) \\ \mathcal{H}_{x_0,A'} \; = \; \{M_y^{x_0}\}_{y\in\pi_X^{-1}(x_0)\cap A'} & \text{of} \quad \pi_X^{-1}(U)\cap A' \end{cases}$$

and whose generating frames fields will be denoted by

$$(w_1^{XZ},\ldots,w_l^{XZ}) \qquad \text{and} \qquad (w_1^{XA'},\ldots,w_l^{XA'})\,.$$

Let now H_Z be the topological trivialization of origin x_0 of the projection π_{XZ} : $T_{XZ}(1) \to X$ obtained by lifting all vector fields $u_1,\ldots,u_l$ of X to the (π_{XZ},ρ_{XZ})-controlled continuous lifting on the leaves of $\mathcal{H}_{x_0,Z}$:

$$\begin{array}{rcl} H_Z \; := H_{x_0,Z} \; : \;\; U\times\pi_{XZ}^{-1}(x_0) \;\equiv\; \mathbb{R}^l\times\pi_{XZ}^{-1}(x_0) & \longrightarrow & T_{XZ}(1)\subseteq\mathbb{R}^n \\ (t_1,\ldots,t_l,z_0) & \longmapsto & \psi_l(t_l,\ldots,\psi_1(t_1,z_0)\ldots) \end{array}$$

where $\forall i\le l$, ψ_i is the flow of the vector field w_i^{XZ} whose foliation $\mathcal{H}_{x_0,Z}$ satisfies the Whitney fibering conjecture for $X<Z$ on U (by Theorem 4).

Remark that the l^{th} vector field w_l^{XZ}, lift on $T_{XZ}(1)$ of the l^{th} coordinate vector field u_l of U, remains <u>un-modified</u> during the construction in Theorem 4 because of its special position in the composition of the flows defining of H_Z. Hence :

$$(*)_{xz} : \qquad w_l^{XZ}(z)=v_l^{XZ}(z)=v_l(z) \qquad \text{for every} \quad z\in T_{XZ}(1)\,.$$

We can write the leaves of the (a)-regular foliation $\mathcal{H}_{x_0,Z}$ contained in $S_{XZ}(1)$ as :

$$\mathcal{H}_{x_0,S_{XZ}(1)} \; := \; \left\{M_{z_0}^{x_0} \; := \; H_Z(U\times\{z_0\})\right\}_{z_0\in L_{XZ}(x_0,1)} \quad \text{with} \quad L_{XZ}(x_0,1):=\pi_{XZ}^{-1}(x_0)\cap S_{XZ}(1).$$

Similarly, by considering the stratum $X\in\Sigma'$, and the corresponding trivialization map $H_{A'}$ of the projection $\pi_{X|A'} : T_X(1)\cap A'\to X$, we can write the leaves of the (a)-regular foliation $\mathcal{H}_{x_0,A'}$ contained in $S'_X(1):=S_X(1)\cap A'$ as :

$$\mathcal{H}_{x_0,S'_X(1)} \; := \; \left\{M_{y_0}^{x_0} \; := \; H_{A'}(U\times\{y_0\})\right\}_{y_0\in L_{XA'}(x_0,1)} \quad \text{with} \quad L_{XA'}(x_0,1):=\pi_X^{-1}(x_0)\cap S'_X(1)\,.$$

We will prove now that the foliations $\mathcal{H}_{x_0,Z}$ and $\mathcal{H}_{x_0,A'}$ can be glued together into a final foliation satisfying the statement of the Theorem.

Let $H_Y := H_{A'|T_{XY}(1}$ and $\mathcal{H}_{x_0,Y} := \mathcal{H}_{x_0,A'|T_{XY}(1)}$ be the natural restrictions to $T_{XY}(1)$ of $H_{A'}$ and $\mathcal{H}_{x_0,A'}$.

To glue together $\mathcal{H}_{x_0,Z}$ and $\mathcal{H}_{x_0,A'}$ we first need to extend $\mathcal{H}_{x_0,Y}$ into a foliation $\mathcal{H}_{x_0,U',Z}$ of an open set $T_{YZ}(1)\cap T_{XZ}(1)$ using the open book structure Theorem 7 ; we need this in order to obtain the property :

$$\lim_{\substack{z\to x \\ z\in T_{YZ}(1)}} T_z\mathcal{H}_{x_0,U',Z} \; = \; T_xX \qquad \text{for every} \quad x\in U\,.$$

By Theorem 5, by considering the stratifications $\mathcal{X}'$, in which $depth_{\Sigma'} X = s-1$, there exists an (a)-regular open book structure of $U < \pi_X^{-1}(U) \cap A'$ over U :

$$\mathcal{W}_{y_0} \;:=\; \left\{ W_{y_0} \;:=\; H_Y(U \times L_{y_0}) \right\}_{y_0 \in L_{XY}(x_0,1)} \quad \text{with} \quad L_{XY}(x_0,1) := \pi_X^{-1}(x_0) \cap S_{XY}(1)$$

where each $L_{y_0} := \gamma_{y_0}(]0,1])$ is a C^∞-arc contained in the fiber $\pi_{XY}^{-1}(x_0)$ and is parametrized by the link $L_{XY}(x_0,1)$ of x_0 in $\pi_{XY}^{-1}(x_0)$.

Such a property also holds for all other strata X_j such that $X < X_j < Y$ so that there exists an (a)-regular open book structure of $U < \pi_X^{-1}(U) \cap A'$ over U :

$$\mathcal{W}_{y_0} \;:=\; \left\{ W_{y_0} \;:=\; H_{A'}(U \times L_{y_0}) \right\}_{y_0 \in L_{XA'}(x_0,1)} \quad \text{with} \quad L_{XA'}(x_0,1) := \pi_X^{-1}(x_0) \cap S'_X(1).$$

As explained in Example 1, for every chart

$$h \;:\; V \subseteq L_{XY}(x_0,1) \longrightarrow \mathbb{R}^{k-l-1} \qquad \text{of the link} \qquad L_{XY}(x_0,1) := \pi_{XY}^{-1}(x_0) \cap S_{XY}(1)$$

the disjoint union of lines $V' = \sqcup_{y_0 \in V} L_{y_0}$ is the domain of a conical chart of $\pi_{XY}^{-1}(x_0)$ (over $\{x_0\}$) by which we obtain a conical chart $\tilde{\varphi}$ of $\pi_{XY}^{-1}(U) \subseteq Y$ over U defined on $U' = H_Y(U \times V')$:

$$\tilde{\varphi} \;:\; U' := H_Y(U \times V') \;\longrightarrow\; U \times]0,1[\times \mathbb{R}^{k-l-1} \qquad \text{where} \qquad H_Y := H_{A'|Y}$$

defines a sub-foliation of $\mathcal{H}_{x_0,A'}$

$$\mathcal{H}_{x_0,U'} \;:=\; \left\{ M^{x_0}_{y_{0,t}} = H_Y(U \times \{y_{0,t}\}) \;\right\}_{y_0 \in L_{XY}(x_0,1),\, t \in]0,1]} \qquad \text{with} \qquad y_{0,t} = \gamma_{y_0}(t)$$

satisfying the properties of the Remark 7. In particular :

a) the first l components of the coordinate k-frame field $(q_1, \ldots, q_k)$ induced by $\tilde{\varphi}$ on U' give exactly the frame field $\left(w_1^{XY}, \ldots, w_l^{XY}\right)$ lifted to the (a)-regular foliation $\mathcal{H}_{x_0,Y}$, where these last are the natural restrictions :

$$w_i^{XY} := {w_i^{XA'}}_{|Y} \qquad \text{and} \qquad \mathcal{H}_{x_0,Y} := \mathcal{H}_{x_0,A'|Y}\,.$$

Thus we can write :

$$(q_1, \ldots, q_k) := \left(w_1^{XY}, \ldots, w_l^{XY}, \ldots, w_k^{XY}\right).$$

b) If $\mathcal{A} = \{h_i : V_i \to \mathbb{R}^{k-l-1}\}_{i \in I}$ is an atlas of $L(x_0,1)$, $\tilde{\mathcal{A}} := \{\tilde{\varphi}_i : U'_i \to \mathbb{R}^k\}_{i \in I}$ is an atlas of $\pi_{XY}^{-1}(U)$ and making the same construction with two different charts $(h_i, V_i), (h_j, V_j)$ of the link $L_{XY}(x_0,1) := \pi_{XY}^{-1}(x_0) \cap S_{XY}(1)$ one obtains open sets (V'_i, U'_i) and (V'_j, U'_j) such that

$$U'_i \cap U'_j \;=\; H_Y(U \times V'_i) \;\cap\; H_Y(U \times V'_j) \;=\; H_Y\big(U \times (V'_i \cap V'_j)\big)$$

and for every $y_{0\,t} \in V'_i \cap V'_j$, $\tilde{\varphi}_i$ and $\tilde{\varphi}_j$ define the same leaf $M^{x_0}_{y_{0,t}} = H_Y(U \times \{y_{0,t}\})$.

Therefore the l-foliations $\mathcal{H}_{x_0,U'_i}$ and $\mathcal{H}_{x_0,U'_j}$ coincide in the intersections where they are generated by the same frame field : $\left(w_1^{iYZ}, \ldots, w_l^{iYZ}\right) \;=\; \left(w_1^{jYZ}, \ldots, w_l^{jYZ}\right)$ and can be extended in a foliation union $\mathcal{H}_{x_0,U'_i} \cup \mathcal{H}_{x_0,U'_j} = \mathcal{H}_{x_0,U' \cup U'_j}$.

One has then the foliation union on the whole of $\pi^{-1}_{XY}(U)$:

$$\mathcal{H}_{x_0,\pi^{-1}_{XY}(U)} \ := \ \bigcup_{i\in I} \mathcal{H}_{x_0,U_i} \, .$$

c) Properties *a)* and *b)* above also hold for every stratum X_j such that $X < X_j < Y$.

Fix $p_0 = y_{0,\frac{1}{2}} \in \pi^{-1}_{XY}(x_0) \subseteq U' \cong \mathbb{R}^k$.

By Theorem 4 and the (c)-regularity of the pair of strata $Y < Z$, and using the coordinate frame field $(w_1^{XY}, \ldots, w_k^{XY})$ induced by $\tilde{\varphi}$ on U', with p_0 as origin, we define a topological trivialization of the projection $\pi_{YZ} : T_{YZ}(1) \to Y$:

$$H_{YZ} \ : \ U' \times \pi^{-1}_{YZ}(p_0) \ \equiv \ \mathbb{R}^k \times \pi^{-1}_{YZ}(p_0) \ \longrightarrow \ \pi^{-1}_{YZ}(U')$$

and a k-foliation of the image $\pi^{-1}_{YZ}(U')$ induced by H_{YZ} defined by :

$$\mathcal{H}_{p_0,Y,Z} \ := \ \Big\{ N_{z_0}^{p_0} \ := \ H_{YZ}(U' \times \{z_0\}) \Big\}_{z_0 \in \pi^{-1}_{YZ}(p_0)}$$

such that :

$$\lim_{z\to y} \Big(w_1^{YZ}(z), \ldots, w_k^{YZ}(z) \Big) \ = \ \Big(w_1^{XY}(y), \ldots, w_k^{XY}(y) \Big) \quad \text{for every} \quad y \in U'$$

which is hence (a)-regular over $U' < Z$:

$$\lim_{\substack{z\to y \\ z\in Z}} T_z N_z^{p_0} \ = \ T_y N_y^{p_0} \ = \ T_y Y \, , \qquad \text{for every} \ \ y \in U'$$

and whose first l-frame fields $(w_1^{YZ}, \ldots, w_l^{YZ})$ are the continuous (π_{YZ}, ρ_{YZ})-controlled lifting on the foliation $\mathcal{H}_{y_0,Y,Z}$ (of $\pi^{-1}_{YZ}(U')$) of the frame fields $(w_1^{XY}, \ldots w_l^{XY})$ (of $\pi^{-1}_{XY}(U)$) lifting of the frame field $(u_1, \ldots u_l)$ (of U).

In particular we also have :

$$(*)_{yz} : \qquad w_l^{YZ}(z) \ = \ v_l(z) \ = \ w_l^{XZ}(z) \, , \qquad \text{for every} \quad z \in T_{YZ}(1).$$

Moreover since

$$U' \ = \ H_Y(U \times V') \ = \ \bigsqcup_{y_{0,t}\in V'} H_Y(U \times \{y_{0,t}\}) \ = \ \bigsqcup_{y_{0,t}\in V'} M_{y_{0,t}}^{x_0} \ ,$$

it follows that each k-leaf of $\mathcal{H}_{y_0,Y,Z}$ generated by the frame field $(w_1^{YZ}, \ldots, w_k^{YZ})$:

$$N_{z_0}^{p_0} \ := \ H_{YZ}(U' \times \{z_0\}) \ = \ \bigsqcup_{y_{0,t}\in V'} H_{YZ}\big(M_{y_{0,t}}^{x_0} \times \{z_0\}\big)$$

is foliated by the family of l-leaves generated by the frame field $(w_1^{YZ}, \ldots, w_l^{YZ})$:

$$\mathcal{H}_{x_0,U',Z} := \Big\{ F_{z_0}^{y_{0,t}} \ := \ H_{YZ}\big(M_{y_{0,t}}^{x_0} \times \{z_0\}\big) \Big\}_{y_{0,t}\in V',\ z_0\in\pi^{-1}_{YZ}(p_0)} \, .$$

Hence, by $\lim_{z \to x} w_i^{YZ}(z) = u_i(x)$ for every $i \leq l$ **[MT]**$_2$, we have for every $x \in U$:

$$\lim_{\substack{z \to x \\ z \in T_{YZ}(1)}} T_z \mathcal{H}_{x_0,U',Z} \;=\; \lim_{\substack{z \to x \\ z \in T_{YZ}(1)}} \left[w_1^{YZ}(z), \ldots, w_l^{YZ}(z) \right] \;=\; \left[u_1(z), \ldots, u_l(z) \right] = T_x X \,.$$

Remark 8. Making the same construction starting with two different charts $(h_i, V_i), (h_j, V_j)$, one obtains open domains of conical charts U'_i and U'_j of Y such that :

1) the two l-foliations $\mathcal{H}_{x_0,U'_i,Z}$ and $\mathcal{H}_{x_0,U'_j,Z}$ coincide in the intersection ;

2)) if the frame fields of V_i and V_j coincide in the intersection $V_i \cap V_j$ then the k-foliations of leaves $N^{p_0}_{z_0}$, $\mathcal{H}_{x_0,U'_i,Z}$ and $\mathcal{H}_{x_0,U'_j,Z}$ coincide in the intersection. $\square$

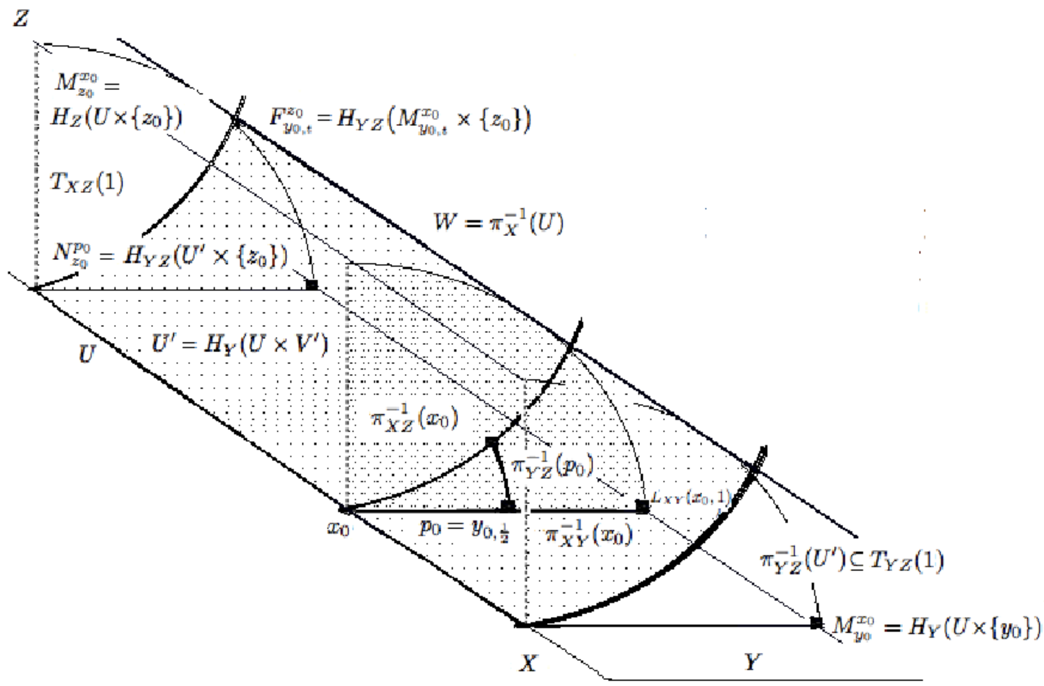

Figure 10

We define a new frame field $(w_1, \ldots, w_l)$ of $\pi_X^{-1}(U)$ by gluing together the frame fields:

$$\left(w_1^{XZ}, \ldots, w_l^{XZ} \right) \qquad \text{and} \qquad \left(w_1^{YZ}, \ldots, w_l^{YZ} \right).$$

To simplify the notations we will denote $\forall\, i = 1, \ldots, l$: $\quad w_i^{XZ} = w_i^1$ and $w_i^{YZ} = w_i^2$.

Note that each l-leaf $M^{x_0}_{z_0}$ of $\mathcal{H}_{x_0,Z}$ meets the fiber $\pi_X^{-1}(x_0)$ in the unique point z_0 and each l-leaf $F^{y_{0,t}}_{z_0}$ of $\mathcal{H}_{x_0,U',Z}$ contained in a k-leaf $N^{p_0}_{z_0}$ of $\mathcal{H}_{p_0,Y,Z}$ meets the fiber $\pi_X^{-1}(x_0)$ in the unique point z_0.

We define $(w_1, \ldots, w_l)$ by decreasing induction on $i = l \geq \ldots \geq 1$.

For $i = l$, define $w_l = w_l^{YZ} = w_l^{XZ}$ (the same vector field).

Let $i = l - 1$, and consider the open covering of $\pi_X^{-1}(x_0) = \pi_{XY}^{-1}(x_0) \cup \pi_{XZ}^{-1}(x_0)$, $\mathcal{O} := \{O_1, O_2\}$ defined by :

$$O_1 \;:=\; \pi_X^{-1}(x_0) \cap \Big(T_{XZ}(1) - \overline{T_{YZ}(1/2)} \Big) \,, \quad O_2 \;:=\; \pi_X^{-1}(x_0) \cap \Big(T_{XZ}(1) \cap T_{YZ}(1) \Big) \,,$$

and let $P_{l-1} := \{\alpha, \beta\}$ be a partition of unity subordinate to the open covering $\mathcal{O}$.

The open covering $\mathcal{O}$ and the partition of unity P_{l-1} on the fiber $\pi_X^{-1}(x_0)$ induce an open covering $\mathcal{O}' := \{O'_1, O'_2\}$ of $\pi_X^{-1}(U)$ where

$$\begin{cases} O'_1 \ := \ H_Z(U \times O_1) \ = \ \bigsqcup_{z_0 \in O_1} H_Z(U \times \{z_0\}) \ = \ \bigsqcup_{z_0 \in O_1} M_{z_0}^{x_0} \\ O'_2 \ := \ H_{YZ}(U' \times O_2) \ = \ \bigsqcup_{z_0 \in O_2} H_{YZ}(M_{y_{0,t}}^{x_0} \times \{z_0\}) \ = \ \bigsqcup_{z_0 \in O_2} F_{z_0}^{y_{0,t}} \end{cases}$$

and a partition of unity $\mathcal{P}_{l-1} := \{\alpha_{l-1}, \beta_{l-1}\}$ of $\pi_X^{-1}(U)$ subordinate to $\mathcal{O}'$ is obtained extending $\{\alpha, \beta\}$ in a constant way along each trajectory of w_l and which is thus *adapted* to $\{O'_1, O'_2\}$ in the sense of the proof of Theorem 2 of §4.

Let w_{l-1} be the vector field defined by :

$$w_{l-1}(z) \ := \ \alpha_{l-1}(z) w_{l-1}^1(z) \ + \ \beta_{l-1}(z) w_{l-1}^2(z)\,.$$

With formally the same calculation as in the proof of Theorem 2 of §4 we find :

$$\begin{aligned} \pi_{X*z}(w_{l-1}(z)) \ &= \ \alpha_{l-1}(y)\pi_{X*z}(w_{l-1}^1(z)) \ + \ \beta_{l-1}(z)\pi_{XY*y}\pi_{YZ*z}(w_{l-1}^2(z)) \\ &= \alpha_{l-1}(z) \cdot u_{l-1}(z) \ + \ \beta_{l-1}(z)\pi_{XY*y}(w_{l-1}^{XY}(z)) \\ &= \alpha_{l-1}(z) \cdot u_{l-1}(z) \ + \ \beta_{l-1}(z) \cdot u_{l-1}(x) = 1 \cdot u_{l-1}(x) = u_{l-1}(x) \end{aligned}$$

and

$$\begin{aligned} \rho_{X*z}(w_{l-1}(z)) &= \alpha_{l-1}(z)\rho_{X*z}(w_{l-1}^1(z)) + \beta_{l-1}(z)\rho_{X*z}(w_{l-1}^2(z)) \\ &= \alpha_{l-1}(z) \cdot 0 + \beta_{l-1}(z) \cdot 0 = 0. \end{aligned}$$

Moreover, the Lie bracket $[w_{l-1}, w_l]$ satisfies :

$$\begin{aligned} [w_{l-1}(y), w_l(y)] \ &= \ [\alpha_{l-1} w_{l-1}^1(y), w_l(y)] \ + \ [\beta_{l-1} w_{l-1}^2(y), w_l(y)] \\ &= \Big(\alpha_{l-1\,*y}(w_l(y)) \cdot w_{l-1}^1(y) \quad + \quad \alpha_{l-1}(y)[w_{l-1}^1(y), w_l(y)]\Big) \ + \\ &+ \Big(\beta_{l-1\,*y}(w_l(y)) \cdot w_{l-1}^2(y) \quad + \quad \beta_{l-1}(y)[w_{l-1}^2(y), w_l(y)]\Big) \ = \ 0 \end{aligned}$$

where $\alpha_{l-1\,*y}(w_l(y)) = \beta_{l-1\,*y}(w_l(y)) = 0$ since α_{l-1} and β_{l-1} are constant along the trajectories of w_l and $[w_{l-1}^1(y), w_l(y)] = [w_{l-1}^2(y), w_l(y)] = 0$ since $(w_1^1, \ldots, w_l^1)$ and $(w_1^2, \ldots, w_l^2)$ are generating frame fields respectively of $\mathcal{H}_{x_0,Z}$ and $\mathcal{H}_{x_0,Y,Z}$ with $w_l^1 = w_l^2 = w_l$.

At this point the definition by induction of w_i for $i < l-1$ is obtained exactly in the same formal way as in Theorem 2 of §4 and this completes the inductive step.

Therefore we obtain a final frame field $(w_1, \ldots, w_l)$ on $W := \pi_X^{-1}(U)$:

$$w_i(z) = \alpha_i(z) w_i^1(z) + \beta_i(z) w_i^2(z) \qquad \text{for every } \ i = 1, \ldots, l$$

by gluing the l-frame fields $(w_1^1, \ldots, w_l^1)$ generating $\mathcal{H}_{x_0,Z}$ together to the l-frame fields $(w_1^2, \ldots, w_l^2)$ generating $\mathcal{H}_{x_0,U',Z}$ such that :

$$(w_1, \ldots, w_l) \ = \ \begin{cases} (w_1^1, \ldots, w_l^1) = (w_1^{XZ}, \ldots, w_l^{XZ}) & \text{on} \quad T_{XZ}(1) - T_{YZ}(1) \\ (w_1^2, \ldots, w_l^2) = (w_1^{YZ}, \ldots, w_l^{YZ}) & \text{on} \quad T_{XZ}(1) \cap T_{YZ}(1/2) \ . \end{cases}$$

The frame field $(w_1,\dots,w_l)$ satisfies the following :

i) $(w_1,\dots,w_l)$ is a (π,ρ)-controlled extension of $(u_1,\dots,u_l)$ of X ;

ii) $(w_1,\dots,w_l)$ is a continuous extension of $(u_1,\dots,u_l)$ of X.

Proof of i). To prove the π-control condition, by induction it is enough to see it for the strata $Z > X$:

$$\begin{aligned}\pi_{XZ*}\big(w_i(z)\big) &= \alpha(z)\pi_{XZ*}\big(w_i^1(z)\big) + \beta(z)\pi_{XY*}\pi_{YZ*}\big(w_i^2(z)\big)\\ &= \alpha(z)w_i^1(\pi_{XZ}(z)) + \beta(z)\pi_{XY*}w_i^2(\pi_{YZ}(z))\\ &= \alpha(z)w_i^1(\pi_{XZ}(z)) + \beta(z)w_i^2\big(\pi_{XY}(\pi_{YZ}(z))\big)\\ &= \big[\alpha(z)+\beta(z)\big]\cdot u_i(\pi_{XZ}(z)) = u_i(\pi_{XZ}(z)).\end{aligned}$$

Similarly, to prove the ρ-control condition we show that :

$$\begin{aligned}\rho_{X*z}(w_i(z)) &= \alpha_{l-1}(z)\rho_{X*z}(w_i^1(z)) + \beta_{l-1}(z)\rho_{X*z}(w_i^2(z))\\ &= \alpha_{l-1}(z)\cdot 0 + \beta_{l-1}(z)\cdot 0 = 0. \qquad \square\end{aligned}$$

Proof of ii). Let X_j be a stratum, $X \le X_j \le Y$ and $x_j \in W \cap T_{XX_j}(1) \subseteq X_j$.

For every $i = 1,\dots,l$ and for every $z \in W$, writing $y = y_{s-1} = \pi_{YZ}(z)$ and $y_j := \pi_{X_jZ}(z)$ for every $j = 0,\dots s$ there are essentially two cases :

Case 1) : $j > 0$, i.e. $X_j > X$.

In this case in a sufficiently small neighbourhood of $x_j \in X_j$ in A, $w_i(z) = w_i^{YZ}(z)$ and so, for z in the neighbourhood $T_{YZ}(\epsilon)$ renamed $T_{YZ}(1)$ (see $(*)$ for its definition), by construction and induction, we have :

$$(**) : \qquad \lim_{z\to x_j} w_i(z) = \lim_{z\to x_j} w_i^{YZ}(z) = w_i^{XA'}(x_j).$$

Thus w_i is a continuous extension of $w_i^{XA'}$ at each $x_j \in X_j$.

Case 2) : $j = 0$, i.e. : $X_j = X_0 = X$ and $x_j = x$.

In this case we can write :

$$\begin{aligned}w_i(z) - w_i^{XA'}(x) &= \alpha(z)\big(w_i^1(z) - w_i^{XA'}(x)\big) + \beta(z)\big(w_i^2(z) - w_i^{XA'}(x)\big)\\ &= \alpha(z)\big(w_i^{XZ}(z) - w_i^{XA'}(x)\big) + \beta(z)\big(w_i^{YZ}(z) - w_i^{XA'}(x)\big)\end{aligned}$$

where as in $(**)$ we have :

$$\lim_{z\to x} \beta(z)\cdot\big(w_i^{YZ}(z) - w_i^{XA'}(x)\big) = 0 \qquad \text{since} \quad \beta(z) \in [0,1]$$

and where, $w_i^{XZ}(z)$ being the continuous lifting on $T_{XZ}(1)$ of $u_i(x) = {w_i^{XA'}}_{|X}(x)$ we have:

$$\lim_{z\to x} \alpha(z)\cdot\big(w_i(z) - w_i^{XA'}(x)\big) = \lim_{z\to x} \alpha(z)\big(w_i^{XZ}(z) - u_i(x)\ \big) = 0\,.$$

Thus w_i is a continuous extension of u_i at each $x \in X$.

We deduce the continuity of each w_i on every stratum X_j of $T_X(1) = \sqcup_{j=1}^{s} T_{XX_j}(1)$:

$$\lim_{\substack{z \to x_j \\ z \in A}} w_i(z) \;=\; w_i^{XA'}(x_j). \quad \square$$

By *ii)* it follows easily that the foliation $\mathcal{F}_{x_0}$ generated by the frame field $(w_1, \ldots, w_l)$:

$$\mathcal{F}_{x_0} \;:=\; \left\{F_z^{x_0}\right\}_{z \in \pi_X^{-1}(x_0)} \quad \text{defined by} \quad F_z^{x_0} \;:=\; [w_1(z), \ldots, w_l(z)]$$

satisfies for every stratum $X_j \,:\, X \leq X_j \leq Z$ and for every $x_j \in U' = W \cap X_j$:

$$\lim_{\substack{z \to x_j \\ z \in A}} T_z F_z^{x_0} \;=\; \lim_{\substack{z \to x_j \\ z \in A}} \big[w_1(z), \ldots, w_l(z)\big] \;=\; \big[w_1^{XA'}(x_j), \ldots, w_l^{XA'}(x_j)\big] \;\subseteq\; T_{x_j} X_j$$

and in particular for every $x \in U = W \cap X$:

$$\lim_{\substack{z \to x \\ z \in A}} T_z F_z^{x_0} \;=\; \lim_{\substack{z \to x \\ z \in A}} \big[w_1(z), \ldots, w_l(z)\big] \;=\; \big[u_1(x), \ldots, u_l(x)\big] = T_x X\,.$$

We conclude then that the foliation $\mathcal{F}_{x_0}$ generated by the frame field $(w_1, \ldots, w_l)$ satisfies all properties in the statement of the Theorem. $\square$

Corollary 4. *Every analytic variety or subanalytic set or definable set in an o-minimal structure satisfies the smooth version of the Whitney fibering conjecture.*

Prof. Since analytic varieties, subanalytic set, and definable sets admit Whitney stratifications (**[Ve], [Hi]** and **[Loi], [NTT]**) and Whitney regularity implies (c)-regularity **[Be]** **[Tr]**$_1$ then the proof follows by Theorem 7. $\square$

We generalize now Theorems 5 and 6 of section 6 to a stratum X of arbitrary depth.

Theorem 8. *Let $\mathcal{X} = (A, \Sigma)$ be a Bekka (c)- (resp. Whitney (b))-regular stratification. Let X be a stratum of $\mathcal{X}$, $x_0 \in X$ and U a domain of a chart near x_0 of X.*

Then $\mathcal{X}$ admits a (c)- (resp. (b)-) regular open book structure $\mathcal{W}_{x_0} = \{W_{z_1}\}_{z_1 \in L(x_0,\epsilon)}$ on $W = \pi_X^{-1}(U)$ over U such that for every stratum $Y \geq X$, $Y \cap W$ is a union of wings, and $\mathcal{W}_{x_1}$ satisfies :

$$(3): \qquad \lim_{\substack{z \to y \\ z \in A}} T_z W_{z_1} \;=\; T_y W_y \;\subseteq\; T_y Y\,. \qquad \textit{for every} \;\; y \in Y \cap W\,.$$

Proof. Let $\mathcal{X}' = (A', \Sigma')$ be the stratification induced by $\mathcal{X}$ on $A' := \pi_X^{-1}(x_0) \cap T_X(1)$:

$$\pi_X^{-1}(x_0) \;=\; \bigsqcup_{X \leq Y} \pi_{XY}^{-1}(x_0)\,.$$

By (c)-regularity, as in Theorems 5 and 6, $\mathcal{X}'$ admits a natural stratified foliaton of lines

$$\mathcal{L}_{x_0} := \Big\{L_{z_1} := \gamma_{z_1}(]0,1[)\Big\}_{z_1 \in L(x_0,1)}$$

satisfying $\{x_0\} \subseteq \overline{L_{z_1}}$ where we suppose as usual (after a change of scale) $\epsilon = 1$.

By (c)-regularity and Theorem 7 there exists a trivialization of $W := \pi_X^{-1}(U)$,

$$\begin{array}{rccc} H: & U \times \pi_X^{-1}(x_0) \equiv \mathbb{R}^l \times \pi_X^{-1}(x_0) & \longrightarrow & W = \pi_X^{-1}(U)\,, \\ & (t_1, \ldots, t_l, z_0) & \longmapsto & z := \phi_l(t_l, \ldots, \phi_1(t_1, z_0) \ldots) \end{array}$$

whose induced "horizontal" foliation

$$\mathcal{H}_{x_0} = \{F_z := H(U \times \{z\}\}_{z \in \pi_X^{-1}(x_0)} \qquad \text{is globally } (a)\text{-regular over } U.$$

Hence, as in Theorems 5 and 6, we define the <u>global</u> family of wings over U :

$$\mathcal{W}_{x_0} \;:=\; \Big\{ W_{z_1} \;=\; H\big(U \times L_{z_1}\big) \;\Big\}_{z_1 \in L(x_0,1)}$$

such that each wing W_{z_1} satisfies :

$$W_{z_1} \;:=\; H\big(U \times L_{z_1}\big) \;\supseteq\; H\big(U \times \{\gamma_{z_1}(s)\}\big) \;=\; F_{\gamma_{z_1}(s)}\,.$$

Then the proofs follow as in Theorems 5 and 6 since, by the <u>global</u> (a)-regularity of the foliation $\mathcal{H}_{x_0}$, this time we can write :

$$\lim_{\substack{z \to x \\ z \in A}} T_z W_{z_1} \;\supseteq\; \lim_{\substack{z \to x \\ z \in A}} T_z F_{\gamma_{z_1}(s)} \;\supseteq\; T_x X\,.$$

This proves (a)-regularity at every $x \in U$ of the strata $U < W_{z_1}$ and this for every wing $W_{z_1} \subseteq W = \sqcup_{X \le Y} \pi_{XY}^{-1}(U)$ and so

$$\mathcal{W}_{x_0} \;:=\; \big\{W_{z_1}\big\}_{z_0 \in L(x_0,1)}$$

is a foliation by wings satisfying the (a)- and (c)-regular open book properties over U.

If $\mathcal{X}$ is (b)-regular, (b^π)-regularity of $U < W_{y_1}$ follows exactly as in Theorem 6.

To show that the foliation of wings $\mathcal{W}_{x_0}$ satisfies the limit property (3), we have to specify more carefully the stratified foliation of lines

$$\mathcal{L}_{x_0} := \Big\{ L_{z_1} := \gamma_{z_1}(]0,1[) \Big\}_{z_1 \in L(x_0,1)}\,.$$

By (c)-regularity, using the theorem of continuous lifting of vector fields $[\mathbf{MT}]_2$ we can obtain the continuity on $\pi_X^{-1}(U) - U = \cup_{X<Y} \pi_{XY}^{-1}(U)$ of the stratified vector field $\gamma'_{z_1}(t) = \{\gamma'_{z_1\,XY}(t)\}_{Y \ge X}$. Hence :

$$(*): \qquad \lim_{z \to y} T_z L_{z_1} = T_y L_{y_1} \qquad \text{with} \qquad y_1 = \pi_{YZ}(z_1) \in Y\,.$$

Let us fix a stratum Y such that $X < Y$ and remark that, with the same notation as Theorem 7, for every stratum $Z > Y$ the neighbourhood $T_{YZ}(1/2)$ is foliated by the family of k-leaves

$$\mathcal{H}_{p_0,Y,Z} \;:=\; \Big\{ N_{z_0}^{p_0} \;:=\; H_{YZ}(U' \times \{z_0\}) \Big\}_{z_0 \in \pi_{YZ}^{-1}(p_0)}$$

which is (a)-regular over $U' = \pi_{XY}^{-1}(U)$: i.e. satisfies the limit property (1) of Theorem 7.

Moreover each leaf $N^{p_0}_{z_0}$ of $\mathcal{H}_{p_0,Y,Z}$ is the continuous lifting of U' and is generated by the frame field $(w^{YZ}_1,\ldots,w^{YZ}_k)$ where (Remark 7) for every

$$z \;=\; \gamma_{z_1}(t) \;\in\; \pi^{-1}_{YZ}(U')\cap T_{YZ}(1/2) \;=\; \pi^{-1}_{XZ}(U)\cap T_{YZ}(1/2)$$

we have :

$$[w^{YZ}_{l+1}(z)] \;=\; [\gamma'_{z_1}(t)] \;=\; T_z L_{z_1} \qquad\text{with}\qquad z_1\in S := S_{XZ}(1)\cap T_{YZ}(1/2)\,.$$

By property $(*)$ above, for every $z_1 \in T_{YZ}(1/2)$ and $y_1 = \pi_{YZ}(z_1)$, the line L_{z_1} is exactly the continuous[(1)] lifting on $T_{YZ}(1/2)$ of the line L_{y_1}, and hence at the level of the wings :

$$(**):\qquad z_1\in L(x_0,1)\cap T_{YZ}(1/2) \quad\Longrightarrow\quad W_{z_1}\text{ is the continuous}^{(1)}\text{ lifting of }W_{y_1}\ .$$

On the other hand for every $z_0 \in S$ one has :

$$\begin{aligned}
N^{p_0}_{z_0} \;:=\; H_{YZ}(U'\times\{z_0\}) \;&=\; \bigsqcup_{\substack{y_1\in L_{XY}(x_0,1)\\ t\in]0,1[}} H_{YZ}\Big(M^{x_0}_{y_1,t}\times\{z_0\}\Big)\\
&= \bigsqcup_{y_1\in L_{XY}(x_0,1)} H_{YZ}\Big(\sqcup_{t\in]0,1[}\,H_Y(U\times\{y_{1,t}\})\times\{z_0\}\Big)\\
&= \bigsqcup_{y_1\in L_{XY}(x_0,1)} H_{YZ}\Big(H_Y\big(U\times(\sqcup_{t\in]0,1[}\{y_{1,t}\})\big)\times\{z_0\}\Big)\\
&= \bigsqcup_{y_1\in L_{XY}(x_0,1)} H_{YZ}\Big(H_Y(U\times L_{y_1})\times\{z_0\}\Big)\\
&= \bigsqcup_{y_1\in L_{XY}(x_0,1)} H_{YZ}\big(W_{y_1}\times\{z_0\}\big) \;=\; \bigsqcup_{z_1\in L_{XZ}(x_0,\frac12)} W_{z_1}\,.
\end{aligned}$$

Here the last equality holds by the property (**) above and since

$$p_0 \;=\; y_{1,\frac12}\in L_{XY}(x_0,1/2) \;=\; \pi^{-1}_{XY}(x_0)\cap S_{XY}(1/2)\,,$$

using the (π,ρ)-control conditions implies :

$$z_0 \;\in\; \pi^{-1}_{YZ}(p_0) \;\subseteq\; \pi^{-1}_{YZ}\Big(\pi^{-1}_{XY}(x_0)\cap S_{XY}\big(1/2\big)\Big) \;=\; \pi^{-1}_{XZ}(x_0)\cap S_{XZ}\big(1/2\big) \;=\; L_{XZ}(x_0,1/2)\,.$$

In conclusion for every $z_0 \in S \subseteq T_{YZ}(1/2)$ each $N^{p_0}_{z_0} \subseteq T_{YZ}(1/2)$ is foliated by the sub-family of wings $\{W_{z_0}\}_{z_0\in L_{XZ}(x_0,1/2)}$ and so for every $y\in Y\cap W$ one has :

$$\begin{aligned}
\lim_{\substack{z\to y\\ z\in A}} T_zW_{z_0} \;&=\; \lim_{\substack{z\to y\\ z\in T_{YZ}(1/2)}} [w^{YZ}_1(z),\ldots,w^{YZ}_{l+1}(z)]\\
&=\; [w^{XY}_1(z),\ldots,w^{XY}_{l+1}(z)] \;=\; T_yW_y \;\subseteq\; T_yY\ . \qquad\square
\end{aligned}$$

(1) One could say *"horizontally-C^1 "* after proving Theorem 10 and Corollary 7 of section 8.

Corollary 5. *Every analytic variety or subanalytic set or definable set in an o-minimal structure admits a stratification Σ in which for every stratum X and every U domain of a chart of X there exists a local (b)-regular open book structure over U.*

Proof. Since analytic varieties, subanalytic sets and respectively definable sets admit Whitney stratifications ([**Ve**], [**Hi**], respectively [**VM**], [**Loi**], [**NTT**]) the proof follows by Theorems 7 and 8. $\square$

8. Horizontally-C^1 and $\mathcal{F}$-semidifferentiable Thom's 1^{st} Isotopy Theorem.

In this section $\mathcal{X} = (A, \Sigma)$ will be a (c)-regular stratification of a closed subset A in a manifold M, X an l-stratum of $\mathcal{X}$, $x_0 \in X$,

$$H \ : \ U_{x_0} \times \pi_X^{-1}(x_0) \to \pi_X^{-1}(U_{x_0}) \, , \qquad H(t_1, \ldots, t_l, y_0) = \phi_l(t_l, \ldots, \phi_1(t_1, y_0)..)$$

the topological trivialization of the projection $\pi_X : T_X(1) \to X$ over a neighbourhood $U_{x_0} \subseteq X$ of X defined by composition of flows $\phi_1, \ldots, \phi_l$ of <u>continuous</u> lifted controlled vector fields $v_1, \ldots, v_l$, and $\mathcal{H} = \{M_y = \{H(U_{x_0} \times \{y_0\})\}_{y_0 \in \pi_X^{-1}(x_0)}$ the (a)-regular foliation defined on $W := \pi_X^{-1}(U_{x_0})$ by H which exists by Theorem 7.

In this section we describe some results, concerning the regularity of the flows of the <u>continuous</u> lifted vector fields to $\mathcal{H}$, consequences of Theorem 7 and which are signifiant because they imply an improvement (stronger than C^0-regularity), of the regularity of the trivialization H which we prove lies between C^0- and C^1-regularity.

Recall that by the Whitney counterexample, *"the four lines family"*, the flows ϕ_i cannot be made in general C^1. We obtain, finally, a horizontally-C^1 version of Thom's 1^{st} Isotopy Theorem for a stratified proper submersion $f : \mathcal{X} \to M$ into a manifold.

These results were initially announced under the hypothesis of the existence of an (a)-regular foliation without proof in $[\mathbf{MT}]_{1,3}$, then proved in $[\mathbf{MT}]_4$. By Theorem 7 they apply to all strata X of a (c)-regular stratification. The proofs are contained in $[\mathbf{MT}]_4$.

8.1. Horizontally-C^1 stratified morphisms and Thom's 1^{st} Isotopy Theorem.

In $[\mathbf{MT}]_{1,3,4}$ we introduce the notions of canonical distributions $\mathcal{D}_X$ associated to each l-stratum X of $\mathcal{X} \subseteq M$ and of horizontally-C^1 stratified controlled maps $f : \mathcal{X} \to \mathcal{X}'$.

A *canonical distribution* $\mathcal{D}_X := \{\mathcal{D}_{XY}\}_{Y \geq X}$ is a continuous l-subbundle of TM, characterized by the property that for each vector field ξ_X defined on X there exists a *canonical* stratified continuous (π_X, ρ_X)-controlled extension to $\mathcal{D}_X$ of ξ_X $[\mathbf{MT}]_{2,3}$.

Definition 11. Let $f : \mathcal{X} \to \mathcal{X}'$ be a stratified morphism between two regular stratifications $\mathcal{X} = (A, \Sigma)$ and $\mathcal{X}' = (A', \Sigma')$ in smooth manifolds M and (resp.) N, X an l-stratum of $\mathcal{X}$ and $x \in X$. For each stratum X of $\mathcal{X}$ let X' be the stratum of $\mathcal{X}'$ containing $f(X)$.

We say that f is *horizontally-C^1 at $x \in X$* if there exists a canonical l-distribution $\mathcal{D}_X = \{\mathcal{D}_{XY}\}_{Y \geq X}$ such that for each stratum $Y > X$ (so $Y' \geq X'$), the restriction $f_{Y*|\mathcal{D}_{XY}} : \mathcal{D}_{XY} \to TY'$ extends continuously the differential $f_{X*} : TX \to TX'$.

That is for every sequence $\{(y_n, v_n)\}_n \subseteq \cup_{y \in Y}\{y\} \times \mathcal{D}_{XY}(y)$:

$$\lim_{n \to \infty} (y_n, v_n) = (x, v) \in TX \quad \Longrightarrow \quad \lim_{n \to \infty} f_{Y*y_n}(v_n) = f_{X*x}(v) \ .$$

This makes sense because by the frontier condition, $X \subseteq \overline{Y} \subseteq M$, and (a)-regularity implies that $TX \subseteq \overline{TY}$ and $TX' \subseteq \overline{TY'}$ in TM and (resp.) TN.

Remark 9. *If the projections of a system of control data of a stratification* (A, Σ) *are* C^1 *then every controlled map* $f : (A, \Sigma) \to M$ *into a manifold* M *is horizontally-*C^1. $\square$

Continuous controlled lifting of vector fields plays an important role in studying horizontally-C^1 regularity. In fact, if a vector field ξ_X is lifted to a stratified continuous (π, ρ)-controlled vector field $\xi = \{\xi_Y\}_{Y \geq X}$ on a neighborhood T_X of X in A, then assuming the existence of an integrable canonical distribution $\mathcal{D}_X$ the lifted flow $\phi = \cup_{Y \geq X} \phi_Y$ on T_X is a horizontally-C^1 extension of ϕ_X ([**MT**]$_4$, Theorem 4).

An arbitrary canonical distribution $\mathcal{D}_X$ is not integrable in general. However, for a stratum X of a (c)-regular stratification $\mathcal{X}$, by Theorem 7 we can consider as canonical distribution $\mathcal{D}_X = T\mathcal{H}$ the distribution tangent to a local (a)-regular foliation and we find:

Corollary 6. *Let* $\mathcal{D}_X = T\mathcal{H}$ *be the canonical distribution tangent to an* (a)*-regular foliation near* $x_0 \in U \subseteq X$, ξ_X *a smooth vector field on* X *and* $\xi = \{\xi_Y\}_{Y \geq X}$ *its continuous controlled lifting tangent to* $\mathcal{H} = \{M_y\}_{y \in W}$.

Then the flow $\phi = \{\phi_Y : Y \to Y\}_{Y \geq X}$ *(to a fixed* $t \in \mathbb{R}$*) of* ξ *is horizontally-*C^1 *on* U.

Proof. Theorem 4 of [**MT**]$_4$. $\square$

With the same hypothesis and notations as in the beginning of section 5 we have:

Corollary 7. *The following properties hold and are equivalent conditions :*

1) There exists a horizontal foliation $\mathcal{H} = \{M_y\}_{y \in W}$ *which is* (a)*-regular on a neighbourhood* U_{x_0} *of* $x_0 \in X$.

2) The topological trivialization homeomorphism of the projection $\pi_X : T_X \to X$,

$$H \ : \ U_{x_0} \times \pi_X^{-1}(x_0) \to \pi_X^{-1}(U_{x_0}) \,, \qquad H(t_1, \ldots, t_l, y_0) = \phi_l(t_l, \ldots, \phi_1(t_1, y_0)..)$$

is horizontally-C^1 *on* U_{x_0}.

3) $\lim\limits_{(t_1,\ldots,t_l,y_0) \to x} H_{*(t_1,\ldots,t_l,y_0)}(E_i) = E_i, \qquad \forall x \in U_{x_0} \equiv \mathbb{R}^l, \ \forall i = 1, \ldots, l \ ;$

4) The controlled liftings $w_1, \ldots, w_l$ *tangent to the foliation* $\mathcal{H} = \{M_y\}_{y \in W}$ *of the standard vector fields* $E_1, \ldots, E_l$ *are continuous on* U_{x_0} *and have horizontally-*C^1 *flows* $\psi_i = \{\psi^t_{iY} : Y \to Y\}_{Y \geq X}$ *on* U_{x_0}.

5) The controlled lifting ξ *tangent to* $\mathcal{H} = \{M_y\}_{y \in W}$ *of every vector field* ξ_X *on* X *is continuous over* U_{x_0} *and has a horizontally-*C^1 *flow* $\psi = \{\psi^t_Y : Y \to Y\}_{Y \geq X}$ *on* U_{x_0}.

Proof. The equivalence of the properties 1), . . . 5) is proved in Theorem 8 in [**MT**]$_4$.

Since by hypothesis $\mathcal{X} = (A, \Sigma)$ is a (c)-regular stratification and $X \in \Sigma$ a stratum of $\mathcal{X}$ such an (a)-regular foliation $\mathcal{H}$ exists by Theorem 7.

Hence the property 1) holds and properties 2), . . . 5) hold too. $\square$

The remark below is elementary :

Remark 10. *The foliation* $\mathcal{H}$ *is* (a)*-regular on* U_{x_0} *if and only if the stratified horizontal projection* π'

$$\pi' : \pi_X^{-1}(U_{x_0}) \longrightarrow \pi_X^{-1}(x_0) \quad , \quad \pi'(M_{y_0}) = y_0$$

satisfies the (a_f) *condition of Thom on* U_{x_0}. $\square$

Definition 12. Let $f = \{f_Y\}_Y : \mathcal{X} \to \mathcal{X}'$ be a stratified morphism $X \in \Sigma$, $x_0 \in X$.

We say that f is *π'-controlled (with respect to the foliations $\mathcal{H} = \{M_y\}_{y \in W}$ of A and resp. $\mathcal{H}' = \{M'_y\}_{y' \in W'}$ of A')* if f sends each leaf of $\mathcal{H}$ into a unique leaf of $\mathcal{H}'$. I.e. :

$$f_Y(M_y) \subseteq M_{y'} \qquad \text{for every } M_y \in \mathcal{H} \ \text{ (where } y' = f_Y(y)).$$

Of course, since $M_y = {\pi'}^{-1}_{XY}(\pi'_{XY}(y))$ this happens if and only if f satisfies the *"horizontal control condition"* :

$$f_Y\big(\pi'^{-1}_{XY}\big(\pi'_{XY}(y)\big)\big) \ \subseteq \ \pi'^{-1}_{X'Y'}\big(\pi'_{X'Y'}\big(f_Y(y)\big)\big)\,, \qquad \forall\, Y \geq X \ \text{ and } \ \forall\, y \in Y.$$

Corollary 6 also holds for such general morphisms :

Theorem 9. *Let $f : \mathcal{X} \to \mathcal{X}'$ be a stratified morphism between two (c)-regular spaces $\mathcal{X}$ et $\mathcal{X}'$. Let $\mathcal{H} = \{M_y\}_{y \in W}$ and $\mathcal{H}' = \{M_{y'}\}_{y' \in W'}$ be two stratified (a)-regular foliations of the neighbourhoods $W = \pi_X^{-1}(U_{x_0})$ of $x_0 \in X$ in A and (resp.) $W' = \pi_{X'}^{-1}(U'_{x'_0})$ of $x'_0 = f(x_0) \in X'$ in A'.*

If $\mathcal{H}$ and $\mathcal{H}'$ are (a)-regular on U_{x_0} and $U'_{x'_0}$ and if $f : \mathcal{X} \to \mathcal{X}'$ sends each leaf of $\mathcal{H}$ into a unique leaf of $\mathcal{H}'$, then f is horizontally-C^1 on U_{x_0}.

Proof. Theorem 9 in $[\mathbf{MT}]_4$.

Theorem 9 above allows us to prove that every (c)-regular stratification admits a horizontally-C^1 topological trivialization near x_0 in X.

We deduce a general version of the Horizontally-C^1 Thom's 1^{st} Isotopy Theorem.

Theorem 10 (Horizontally-C^1 Thom's 1^{st} Isotopy Theorem).

Let $\mathcal{X} = (A, \Sigma)$ be a (c)-regular stratification, $X \in \Sigma$ a stratum of $\mathcal{X}$ and $\mathcal{H} = \{M_y\}_{y \in W}$ an (a)-regular foliation on U_{x_0} of a neighbourhood $W = \pi_X^{-1}(U_{x_0})$ of x_0 in A which exists by Theorem 7.

Let $f : (A, \Sigma) \to M$ be a stratified proper submersion into a smooth m-manifold M.

For every $m_0 \in M$, and for every domain of a chart $U_{m_0} \equiv \mathbb{R}^m$ of M near m_0, the stratified homeomorphism of the topological trivialisation of f

$$H \ : \ U_{m_0} \times f^{-1}(m_0) \to f^{-1}(U_{m_0})\,, \quad H(t_1, \ldots, t_m, a_0) = \phi_m(t_m, \ldots \phi_1(t_1, a_0))..)$$

is horizontally-C^1 on $U_{m_0} \times \big[f^{-1}(m_0) \cap U_{x_0}\big]$, and its inverse stratified homeomorphism:

$$G \ : \ f^{-1}(U_{m_0}) \to U_{m_0} \times f^{-1}(m_0)\,, \quad G(a) = \Big(f(a), \phi_1(-t_1, \ldots \phi_m(-t_m, a) \ldots)\Big)$$

is horizontally-C^1 on $f^{-1}(U_{m_0}) \cap U_{x_0}$.

Above $f(a) := (t_1, \ldots, t_m)$ and for all $i = 1, \ldots, m$, $\phi_1, \ldots, \phi_m$ are the flows of the continuous controlled lifted vector fields $v_1, \ldots, v_m$, such that $f_*(v_i) = E_i$, on $f^{-1}(U_{m_0})$ of the standard vector fields $E_1, \ldots, E_m \in \mathbb{R}^m \equiv U_{m_0}$.

Proof. Theorem 10 in $[\mathbf{MT}]_4$. $\square$

Corollary 8. *The topological trivialization K of the projection $\pi_X : T_X(1) \to X$ corresponding to the continuous, controlled, integrable frame field $(w_1, \ldots, w_l)$ constructed in Theorem 7, is horizontally-C^1 on each stratum of $U \times \big(\sqcup_{X \leq Y} \pi_{XY}^{-1}(x_0)\big)$ and its inverse stratified homeomorphism K^{-1} is horizontally-C^1 on each stratum $\pi_{XY}^{-1}(U)$ of the stratification $W = \pi_{XY}^{-1}(U) = \sqcup_{X \leq Y} \pi_{XY}^{-1}(U)$.*

Proof. It follows by Theorem 10 applied to the projection $\pi_X : T_X(1) \to X$. $\square$

8.2. $\mathcal{F}$-semidifferentiable stratified morphisms and Thom's 1st Isotopy Theorem.

In this section we generalize the horizontally-C^1 regularity of section 8.1 through the notion of $\mathcal{F}$-semidiferentiability, a finer regularity condition for stratified morphisms.

We saw in §8.1 that the (a)-regularity over a neighbourhood U_{x_0} of x_0 in X, of a foliation $\mathcal{H} = \{M_z\}_{z\in W}$ of $W = \pi_X^{-1}(U_{x_0})$ implies the horizontally-C^1 regularity over U_{x_0} of the stratified flows of continuous lifting of vector fields and of the topological trivialization maps. In a similar way we see here that the (a)-regularity of $\mathcal{H}$ on the whole of W implies, for these stratified morphisms, an analoguous and more complete regularity :

$$\lim_{z\to y'} f_{Z*z|T_zM_z} = f_{Y*y|T_yM_y} \, .$$

The notion of $\mathcal{F}$-semidifférentiability below refines horizontally-C^1 regularity.

Definition 13. Let $\mathcal{F} = \{F_z\}_z$ be an (a)-regular stratified $C^{1,0}$ l-foliation of an open set U of A, Y a stratum of $\mathcal{X}$ and $y \in Y$.

We say a morphism $f = \{f_Z\}_{Z\in\Sigma} : \mathcal{X} \to \mathcal{X}'$ is *$\mathcal{F}$-semidifférentiable at y* iff for every $(y, v) \in TY$ and sequence $\{(z_n, v_n)\} \subseteq T_{z_n}\mathcal{F}$, with Z_n the stratum containing z_n we have :

$$\lim_n (z_n, v_n) \;=\; (y, v) \qquad \Longrightarrow \qquad \lim_n f_{Z_n*z_n}(v_n) \;=\; f_{Y*y}(v).$$

That is the differentials of $f_{|F_{z_n}}$ must converge to the differential of $f_{|F_y}$.

In an obvious way one defines the $\mathcal{F}$-semidifférentiability on a stratum X (or on $X\cap U$) and on $\mathcal{X}$ (or on U).

Remark 11. *Let $f : \mathcal{X} \to \mathcal{X}'$ be a stratified morphism, X a l-stratum of $\mathcal{X}$, $l = \dim \mathcal{F}$. Then $f : \mathcal{X} \to \mathcal{X}'$ is $\mathcal{F}$-semidifférentiable at $x \in X$ iff f is horizontally-C^1 at x (with respect to the canonical distribution $\mathcal{D} = \mathcal{D}(z) = T_z\mathcal{F}$.* $\square$

The analogues of the results of section 8.1 hold again for $\mathcal{F}$-semidifférentiability.

As in Corollary 7 we have :

Corollary 9. *Let $\mathcal{X} = (A, \Sigma)$ be a (c)-regular stratification $X \in \Sigma$ a stratum of $\mathcal{X}$, $x_0 \in X$ and U_{x_0} a neighbourhood of x_0 in X.*

The following properties hold and are equivalent conditions :

1) There exists a horizontal foliation $\mathcal{H} = \{M_y\}_{y\in W}$ which is (a)-regular on a neighbourhood $W = \pi_X^{-1}(U_{x_0})$ of x_0 in A.

2) The topological trivialization homeomorphism of the projection $\pi_X : T_X \to X$,

$$H \;:\; U_{x_0} \times \pi_X^{-1}(x_0) \to \pi_X^{-1}(U_{x_0}) \, , \qquad H(t_1, \ldots, t_l, y_0) = \phi_l(t_l, \ldots, \phi_1(t_1, y_0)..)$$

is $\mathcal{F}$-semidifferentiable on W.

3) $\displaystyle\lim_{(t_1,\ldots,t_l,z_0)\to y} H_{*(t_1,\ldots,t_l,z_0)}(E_i) = w_i(y), \qquad \forall y \in Y \subseteq W \equiv \mathbb{R}^l, \ \forall i = 1, \ldots, l \ ;$

4) The controlled liftings $w_1, \ldots, w_l$ tangent to the foliation $\mathcal{H} = \{M_y\}_{y\in W}$ of the standard vector fields $E_1, \ldots, E_l$ are continuous on W and have $\mathcal{F}$-semidifferentiable flows $\psi_i = \{\psi^t_{iY} : Y \to Y\}_{Y\geq X}$ on W.

5) The controlled lifting ξ tangent to $\mathcal{H} = \{M_y\}_{y\in W}$ of every vector field ξ_X on X is continuous on W and has an $\mathcal{F}$-semidifferentiable flow $\psi = \{\psi^t_Y : Y \to Y\}_{Y\geq X}$ on W.

Proof. Similar to the proof of Corollary 7. □

As in Theorem 8 we have :

Theorem 11. *Let $f : \mathcal{X} \to \mathcal{X}'$ be a stratified morphism between two (c)-regular spaces $\mathcal{X}$ et $\mathcal{X}'$. Let $\mathcal{H} = \{M_y\}_{y \in W}$ and $\mathcal{H}' = \{M_{y'}\}_{y' \in W'}$ be two stratified (a)-regular foliations of the neighbourhoods $W = \pi_X^{-1}(U_{x_0})$ of $x_0 \in X$ in A and (resp.) $W' = \pi_{X'}^{-1}(U'_{x'_0})$ of $x'_0 = f(x_0) \in X'$ in A'.*

If $\mathcal{H}$ and $\mathcal{H}'$ are (a)-regular on W and W' and if $f : \mathcal{X} \to \mathcal{X}'$ sends each leaf of $\mathcal{H}$ into a unique leaf of $\mathcal{H}'$ then f is $\mathcal{H}$-semidifferentiable on W.

Proof. Theorem 10 in **[MT]**$_4$.

As in Theorem 9 one also has :

Theorem 12. ($\mathcal{H}$-semidifferentiable Thom's 1^{st} Isotopy Theorem).

Let $\mathcal{X} = (A, \Sigma)$ be a (c)-regular stratification, $X \in \Sigma$ a stratum of $\mathcal{X}$, $x_0 \in X$, U_{x_0} a domaine of a chart near x_0 in X and $\mathcal{H} = \{M_y\}_{y \in W}$ an (a)-regular foliation of the neighbourhood $W = \pi_X^{-1}(U_{x_0})$ of x_0 in A which exists by Theorem 7.

Let $f : (A, \Sigma) \to M$ be a stratified proper submersion into a smooth m-manifold M.

For every $m_0 \in M$, and for every domain of a chart $U_{m_0} \equiv \mathbb{R}^m$ of M near m_0, the stratified homeomorphism of topological trivialisation of f

$$H \ : \ U_{m_0} \times f^{-1}(m_0) \to f^{-1}(U_{m_0}) \ , \quad H(t_1, \dots, t_m, a_0) = \phi_m(t_m, \dots \phi_1(t_1, a_0))..)$$

is $\mathcal{H}$-semidifferentiable on $U_{m_0} \times \big[f^{-1}(m_0) \cap U_{x_0} \big]$, and its inverse stratified homeomorphism:

$$G \ : \ f^{-1}(U_{m_0}) \to U_{m_0} \times f^{-1}(m_0) \ , \quad G(a) = \Big(f(a), \phi_1(-t_1, \dots \phi_m(-t_m, a) \dots) \Big)$$

is $\mathcal{H}$-semidifferentiable on $f^{-1}(U_{m_0}) \cap U_{x_0}$.

Above $f(a) := (t_1, \dots, t_m)$ and for all $i = 1, \dots, m$, $\phi_1, \dots, \phi_m$ are the flows of the continuous controlled lifted vector fields $v_1, \dots, v_m$, such that $f_*(v_i) = E_i$, on $f^{-1}(U_{m_0})$ of the standard vector fields $E_1, \dots, E_m \in \mathbb{R}^m \equiv U_{m_0}$.

Proof. Theorem 10 in **[MT]**$_4$. □

As in corollary 8 we have :

Corollary 10. *The topological trivialization K of the projection $\pi_X : T_X(1) \to X$ corresponding to the continuous, controlled, integrable frame field $(w_1, \dots, w_l)$ constructed in Theorem 7,*

$$K \ : \quad U \times \Big(\bigsqcup_{X \leq Y} \pi_{XY}^{-1}(x_0) \Big) \quad \longrightarrow \quad \pi_X^{-1}(U) \ = \ \bigsqcup_{X \leq Y} \pi_{XY}^{-1}(U)$$

is $\mathcal{F}$-semidifferentiable, $\mathcal{F} = \{U \times \{z\}\}_{z \in \pi_X^{-1}(x_0)}$, at each point of $U \times \big(\sqcup_{X \leq Y} \pi_{XY}^{-1}(x_0) \big)$ and its inverse stratified homeomorphism K^{-1} is $\mathcal{F}_{x_0}$-semidifferentiable at each point of the stratification $W = \pi_X^{-1}(U) = \sqcup_{X \leq Y} \pi_{XY}^{-1}(U)$.

Proof. An immediate consequence of Theorem 12. □

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